Elementary Algebra

(Indices, Surds, Rationalising factors, Polynomials)

Introduction

Laws of Indices

	Rule	Example
1.	a +a +a +… n times = na	7 + 7 + 7 + 7 = 4 x 7 = 28
2.	a.a.a.a…..n times = a^n	7 x 7 x 7 x 7 x 7 x 7 x 7 x 7 x 7 = 7^9
3.	$a^m \times a^n = a^{m+n}$	$5^5 \times 5^6 = 5^{11}$
4.	$\frac{\text{a.a…..n times}}{\text{a.a…..m times}} = \text{a.a…..(n-m) times} = a^{n-m}$	$\frac{8^8}{8^3} = 8^5$
5.	$(a^m)^n = a^{mxn}$	$(5^5)^3 = 5^{5x3} = 5^{15}$
6.	$a^{-n} = \frac{1}{a^n}$	$2^{-8} = \frac{1}{2^8}$
7.	$(ab)^m = a^m b^m$	$4^5 \times 5^5 = (4 \times 5)^5 = (20)^5$
8.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{6^5}{3^5} = \left(\frac{6}{3}\right)^5 = 2^5 = 32$
9.	$a^{\frac{1}{n}} = \sqrt[n]{a}$	$(16)^{\frac{1}{4}} = \sqrt[4]{16} = 2$
10.	$a^{p/q} = \sqrt[q]{a^p}$ where p & q $\in$ R and q $\neq$ 0	$27^{\frac{2}{3}} = \sqrt[3]{(27)^2} = 9$
11.	If $a^m = a^n$ and a $\neq$ – 1, 0, 1, then m = n.	$3^a = 9^b \Rightarrow 3^a = 3^{2b}$ $\Rightarrow$ a = 2b
12.	If $a^m = b^m$, m $\neq$ 0, a = b, if m is odd. a = $\pm$ b, if m is even.	$x^3 = 2^3 \Rightarrow x = 2 \Rightarrow x^2 = 2^2 \Rightarrow x = \pm 2$

Surds or Radicals

In this section, we shall introduce the concepts of a surd and its order. But, let us first understand the concept of positive n^{th} root of a real number.

Positive n^{th} Root of a real number:

Let ‘a’ be a real number and ‘n’ be a positive integer. Then a number which when rose to the power n gives ‘a’ is called the n^{th} root of ‘a’ and it is written as $\sqrt[n]{a}$ or $a^{1/n}$.

Thus n^{th} root of a real number ‘a’ is a real number ‘b’ such that b^n = a. The real number ‘b’ is denoted by $a^{1/n}$ or $\sqrt[n]{a}$.

The cube root of 2 is the real number whose cube is 2. The cube root of 2 is denoted by the symbol $2^{1/3}$ or $\sqrt[3]{2}$. The fourth root of 81 is the real number 3, because 3^4 = 81. The fourth root of 81 is denoted by $\sqrt[4]{81}$.

TIP

It is clear from the definition that every surd is an irrational number but every irrational number is not a surd.

If n is a positive integer and a is a real number, then $\sqrt[n]{a}$ is not a surd if $\sqrt[n]{a}$ is rational.

Surds or Radicals:

If **a** is a rational number and **n** is a positive integer such that the n^{th} root of **a** i.e. $a^{1/n}$ or $\sqrt[n]{a}$ is an irrational number, then $a^{1/n}$ is called a **surd** or **radical** of order **n** and **a** is called the ***radicand***. E.g.:- Since 5 is a rational number and $5^{\frac{1}{2}}$ is a irrational number, hence it is a surd but on the other hand 16 is a rational number and $(16)^{\frac{1}{4}}$ is not a irrational number. Therefore, it is not a surd.

Law of Radicals

As we have seen that surds can be expressed with fractional exponents (indices), the laws of indices are therefore applicable to surds also. The laws of radicals are very useful to simplify a given radical or to reduce two given radicals to the same form.

Ist Law: *For any positive integer 'n' and a positive rational number 'a'*

$$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a.$$

IInd Law: *If n is a positive integer and a, b are rational numbers, then*

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$$

IIIrd Law: *If n is a positive integer and a, b are rational numbers, then*

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

IVth Law: *If m, n are positive integers and a is a positive rational number, then*

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$$

Vth Law: *If m, n are positive integers and a is a positive rational number, then*

$$\sqrt[n]{\sqrt[m]{(a^p)^m}} = \sqrt[n]{a^p} = \sqrt[mn]{a^{pm}}$$

Surds in simplest form:

A surd is said to be in its simplest form if it has:-

(i) No fraction under the radical sign.

(ii) No factor which is n^{th} power of a rational number under the radical sign of index n.

(iii) The smallest possible index of this radical i.e., the order of the surd is the smallest possible order.

Rationalising Factor

If the product of two surds is a rational number, then each one of them is called the rationalising factor (R.F) of the other.

Let us understand with the help of few examples:

⇨ $\sqrt{5}$ is a rationalising factor of $3\sqrt{5}$, because $3\sqrt{5} \times \sqrt{5} = 3 \times 5 = 15$ which is a rational number. Also, $2\sqrt{5}$ is a rationalising factor of $\sqrt{5}$, because $\sqrt{5} \times 2\sqrt{5} = 2 \times 5 = 10$, which is a rational number. In fact, $k\sqrt{5}$ is a rationalising factor of $\sqrt{5}$, where k is any non-zero rational number.

TIP

Rationalising factors of $a \pm \sqrt{b}$ and $\sqrt{a} \mp \sqrt{b}$ are $a \mp \sqrt{b}$ and $\sqrt{a} \pm \sqrt{b}$ resp.

⇨ Since $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$, therefore $\sqrt{3}-\sqrt{2}$ is a R.F. of $\sqrt{3}+\sqrt{2}$ and vice-versa.

Conjugate Surds:

Two binomial surds which differ only in sign (+ or –) between the terms connecting them, are called ***conjugate surds***. e.g. $2+\sqrt{3}$ and $2-\sqrt{3}$ are conjugate surds.

TIP

$a \pm \sqrt{b}$ and $a \mp \sqrt{b}$ are conjugate binomial surds.

Ex.1 Which of the following surds is the greatest?

(1) $2^{1/2}$ (2) $3^{1/3}$ (3) $4^{1/4}$ (4) $6^{1/6}$ (5) $12^{1/12}$

Sol. Given surds are

$2^{1/2}, 3^{1/3}, 4^{1/4}, 6^{1/6}, 12^{1/12}$

Raise the power of each surd to 12 then the surds will become.

$(2^{1/2})^{12}, (3^{1/3})^{12}, (4^{1/4})^{12}, (6^{1/6})^{12}, (12^{1/12})^{12}$

$2^6, 3^4\ 4^3\ 6^2\ 1^2$

3^4 is the great of all.

So $3^{1/3}$ is greatest. **Answer: (2)**

Ex.2 The value of $\sqrt{7+2\sqrt{12}} + \sqrt{7-4\sqrt{3}}$ is

Sol. $\sqrt{7+2\sqrt{12}} = \sqrt{4+3+2\sqrt{4\times 3}}$

$= \sqrt{\left(\sqrt{4}\right)^2 + \left(\sqrt{3}\right)^2 + 2\sqrt{4\times\sqrt{3}}}$

$= \sqrt{4} + \sqrt{3} = 2 + \sqrt{3}$

Similarly $\sqrt{7-4\sqrt{3}} = \sqrt{7-2\sqrt{12}} = 2-\sqrt{3}$

Answer = $2+\sqrt{3}+2-\sqrt{3} = 4$.

Ex.3 If $\dfrac{(0.09)^{3/2}}{(0.4)^3} \times \dfrac{(64)^{\frac{2x-3}{4}}}{6^3} = 1$. What is the value of x?

Sol. Given $\dfrac{\left(9\times 10^{-2}\right)^{3/2}}{\left(4\times 10^{-1}\right)^3} \times \dfrac{\left(2^6\right)^{\frac{2x-3}{4}}}{3^3\times 2^3} = 1$

$\Rightarrow \dfrac{3^3\times 10^{-3}}{4^3\times 10^{-3}} \times \dfrac{2^{\frac{6x-9}{2}}}{3^3\times 2^3} = 1.$

$\Rightarrow 2^{\frac{6x-9}{2}} = 2^9$

$\therefore \dfrac{6x-9}{2} = 9$

$\Rightarrow x = \dfrac{9}{2}$

Ex.4 If $\sqrt{a} = \sqrt[3]{b} = \sqrt[4]{c}$. And abc = 8, what is the value of a?

Sol. It is given $a^{1/2} = b^{1/3} = c^{1/4} = k$ (assume)

Therefore, $a = k^2$, $b = k^3$, $c = k^4$

So $abc = k^9 = 8 \quad \Rightarrow k^3 = 2$

$\Rightarrow k = \sqrt[3]{2}$

$\therefore a = k^2 = \sqrt[3]{4}$

Ex.5 $t_n = \frac{1}{\sqrt{n}+\sqrt{n-1}}$, n ≥ 2.Then what is the value of $t_2 + t_3 + t_4 + \ldots + t_{81}$:

Sol. $t_2 + t_3 + t_4 + \ldots + t_{81} = \frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{81}+\sqrt{80}}$.

By Rationalizing,

$= \frac{1}{2}(\sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \ldots + \sqrt{80} - \sqrt{79} + \sqrt{81} - \sqrt{80}) = \frac{1}{2}(\sqrt{81} - \sqrt{\mathbf{1}}) = \frac{1}{2}(9 - 1) = 4.$

Logarithms & its properties

If $a^x = b$, we can write $a = b^x$, that means we can write b in terms of a & x, and a in terms of b & x.

Suppose, if we want to write x in terms of a & b, that can be written as $x = \log_a b$.

So if $a^x = b$, then $x = \log_a b$.

(It can be read as log b to the base a) (where a is called the base)

Properties of logarithm

Logarithm is not defined for zero and for negative numbers.

1. $\log_a 0$ = not defined
2. $\log_a 1 = 0$
3. $\log_a m + \log_a n = \log_a mn$
4. $\log_a m^n = n \log_a m$
5. $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$
6. $\log_b a = \frac{\log_c a}{\log_c b}$
7. $\log_a a = 1$
8. $\log_b a = \frac{1}{\log_a b}$
9. $a^{\log_a m} = m$
10. $\log_{a^n} m = \frac{1}{n}.\log_a m$

Ex.6 If log (m + n) = log m + log n ; then m = ?

(1) $\frac{m}{(n-1)}$ **(2)** $\frac{n}{(m-1)}$ **(3)** $\frac{n}{(n-1)}$ **(4)** $\frac{n^2}{(n-1)}$ **(5) None of these**

Sol. log (m + n) = log m + log n

$\Rightarrow$ log (m + n) = log mn

$\Rightarrow$ m + n = mn or mn – m = n

$\Rightarrow$ m (n – 1) = n

$\therefore m = \frac{n}{n-1}$. **Answer: (3)**

Ex.7 If $2^{2x+3} = 6^{x-1}$, x equals to:

(1) $\frac{4\log 2 + \log 3}{\log 3 - \log 2}$ **(2)** $\frac{3\log 2 + 2\log 3}{\log 3 - 2\log 2}$ **(3)** $\frac{4\log 2 + \log 3}{\log 3 + \log 2}$

(4) $\frac{3\log 2 + 2\log 3}{\log 3 + 2\log 2}$ **(5) None of these**

Sol. Taking log of both sides, we get

(2x + 3) log 2 = (x – 1) log 6

$\Rightarrow$ 2x log 2 + 3 log 2 = (x – 1) log (2 $\times$ 3) = (x – 1) (log2 + log3)

$\Rightarrow$ 2x log 2 + 3 log 2 = x log 2 + x log 3 – log 2 – log 3

$\Rightarrow$ x log 2 – x log 3 = – 4 log 2 – log 3

$\Rightarrow$ – xlog2 + xlog3 = 4 log 2+ log 3

$\Rightarrow$ x(– log2 + log3) = 4log2 + log 3

$\therefore$ x = $\frac{4\log 2 + \log 3}{\log 3 - \log 2}$. **Answer: (1)**

Ex.8 Evaluate: $\frac{\log_a n}{\log_{ab} n}$.

(1) 1 + $\log_a b$ **(2) 1 + $\log_b a$** **(3) 0** **(4) 1** **(5) None of these**

Sol. $\frac{\log_a n}{\log_{ab} n} = \frac{\log n}{\log a} \times \frac{\log ab}{\log n} = \frac{\log ab}{\log a} = \frac{\log a + \log b}{\log a}$

= 1 + $\log_a b$ **Answer: (1)**

Ex.9 The sum of the series log3 + log $\frac{9}{4}$ + log $\frac{27}{16}$ + log $\frac{81}{64}$ +……..upto 10 terms is

(1) 55log3 – 110log2 **(2) 5(11log3 – 9log4)** **(3) 55log3 – 11log2**

(4) 5(9log3-11log4) **(5) None of these**

Sol. log3 + log9/4 + log 27/16 + log 81/64 + …. 10 terms

= [log3 + $\log(3)^2$ + … + $\log(3)^{10}$] – [log4 + log 4^2 + log 4^3 + ---- $\log 4^9$]

= (1 + 2 + … + 10) log3 – (1 + 2 +… + 9) log4 = 55log3 – 45log4.

Answer: (2)

Ex.10 If $\log_{10}x - \log_{10}\sqrt{x}$ = 2, $\log_x 10$ then x equals to

Sol. $\log_{10}x - \frac{1}{2}\log_{10}x = 2\log_x 10$

or $\frac{1}{2}\log_{10}x = 2\log_x 10$

$\frac{1}{2}\log_{10}x = \frac{2}{\log_{10} x}$

$\Rightarrow (\log_{10}x)^2 = 4$ or $\log_{10}x = \pm 2$.

If $\log_{10}x = 2$, x = 10^2 = 100

If $\log_{10}x = -2$, x = $10^{-2} = \frac{1}{100}$ Therefore, **x = 100 or** $\frac{1}{100}$

Ex.11 log $(x^3 + 5) = 3 \log (x + 2)$. Then x =?

Sol. $\log (x^3 + 5) = 3 \log (x + 2)$

$\Rightarrow x^3 + 5 = (x + 2)^3$

$\Rightarrow x^3 + 5 = x^3 + 8 + 6x^2 + 12x$

$\Rightarrow 6x^2 + 12x + 3 = 0 \Rightarrow 2x^2 + 4x + 1 = 0.$

$\Rightarrow x = \frac{-4 \pm \sqrt{16-8}}{4} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-2 \pm \sqrt{2}}{2}.$

Ex.12 If $a^2 = b^3 = c^5 = d^6$, then $\log_d (abc)$ =

Sol. $a^2 = b^3 = c^5 = d^6$

$\Rightarrow a = d^{6/2} = d^3$

$\Rightarrow b = d^{6/3} = d^2$

$\Rightarrow c = d^{6/5}$

$\log_d (abc) = \log_d (d^3 \times d^2 \times d^{6/5}) = \log_d d^{31/5}$

$= \frac{31}{5} \log_d d = \frac{31}{5}.$

Ex.13 If $f(x) = \frac{3^{1+\log x}}{x^{\log 3}}$, then f(1994)

Sol. Given $f(x) = \frac{3^{1+\log x}}{x^{\log 3}}$

Taking log both sides: $\log f(x) = \log (3)^{(1+\log x)} - \log (x)^{(\log 3)}$

$\Rightarrow \log f(x) = (1 + \log x) \log 3 - \log 3 \log x = \log 3$

$\Rightarrow f(x) = 3$, for all values of x. which is a constant.

$\therefore$ f(1994) = 3.