Function

Definition

Suppose, if we take any system, the output will be a function of input. That means a function is a relation between input and output.

For Example, there is a system, which finds the square of the given input. That means the output is a square of the given input.

This can be represented by output= $(\text{input})^2$

Or $f(x) = x^2$. Where x is input and f(x) is out put. Here f is called the function of x, which is defined as $f(x) = x^2$.

So, if $f(x) = x^2$, $f(1) = 1^2 = 1$, and $f(2) = 2^2 = 4$.

In general, if $f(x) = x^2$, $f(a) = a^2$.

Ex.1 ***(a)*** ***If $f(x) = 2x^2 - 2x + 1$, find $f(-1)$.***

(b) ***If $f(t) = 3t - 1$, find $f(a^2)$***

Sol. (a) We substitute – 1 in place of x.

$f(-1) = 2(-1)^2 - 2(-1) + 1 = 5$

(b) We substitute a^2 in place t.

$f(a^2) = 3(a^2) - 1 = 3a^2 - 1$.

Odd and Even Functions

Odd function:

A function f is said to be odd if it changes sign when the sign of the variable is changed. i.e. If $f(-x) = -f(x)$.

For example: $f(x) = \sin x$; $0 \le x \le 2\pi$ is a odd function.

Even function:

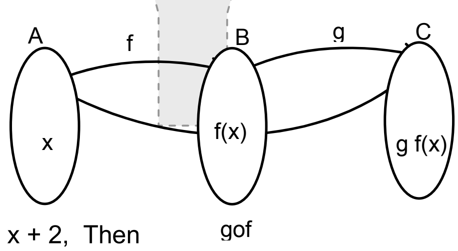
A function f is said to be an even function if it doesn't change sign when the sign of the variable is changed. i.e. if $f(-x) = f(x)$.For example $f(x) = x^4 + x^2$ and $g(x) = \cos x$ are even functions.

<u>NOTE:</u> There are many functions which are neither odd nor even i.e it is not nessesary for a function to be either even of to be odd. E.g: $g(x) = 3x^3 + 4x^2 - 9$ is a function in x which is neither even nor odd.

Composite Functions

A composite function is the function of another function. If f is a function from A in to B and g is a function from B in to C, then their composite function denoted by (g o f) is a function from A in to C defined by

(g o f) (x) = g [f(x)]



e.g. if $f(x) = 2x$, and $g(x) = x + 2$, Then

$(gof)(x) = g[f(x)] = g(2x) = 2x + 2$

$(fog)(x) = f[g(x)] = f(x + 2) = 2(x + 2) = 2x + 4$

<u>FACT :</u> This shows that it is not nessesary that **(fog)(x) = (g of)(x).**

Ex.2 ***Let a function $f_{n+1}(x) = f_n(x) + 3$. If $f_2(2) = 4$. Find the value of $f_6(2)$.***

Sol. We have, $f_{n+1}(x) = f_n(x) + 3$

So, $f_3(2) = f_2(2) + 3 = 7$

$f_4(2) = f_3(2) + 3 = 10$

$f_5(2) = f_4(2) + 3 = 13$

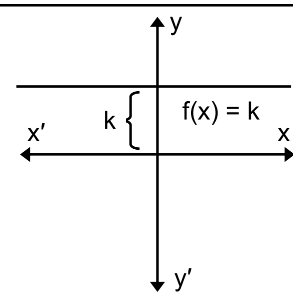
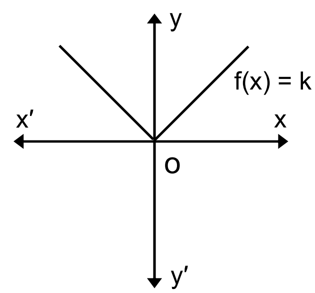
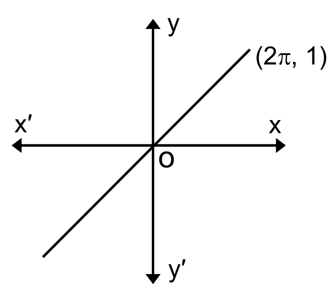
$f_6(2) = f_5(2) + 3 = 16$

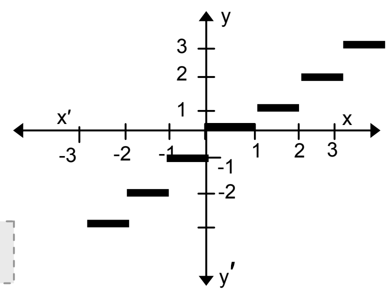
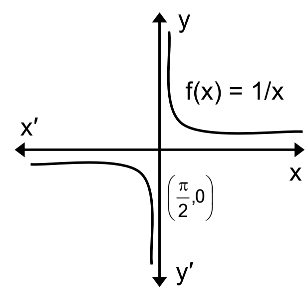
Alternate Method:

Since the function is increasing with constant value.

So, $f_6(2) = = f_2(2) + 3(6 - 2) = 4 + 12 = 16$

Graphs of some special functions

TYPES OF FUNCTIONS	GRAPHS
Constant Function f(x) = k **Domain:** R **Range:** k	
Modulus Function f(x) = $\|x\|$ or f(x) = – x when x < 0 x when x ≥ 0 **Domain:** R **Range:** R^+	
Identity function f(x) = x **Domain:** R **Range:** R	

Greatest Integer Function or Step Function f(x) = [x] **Domain:** R **Range:** Integer	
Reciprocal function f(x) = $\frac{1}{x}$ **Domain:** R – {0} **Range:** R – {0}	

Important: If f and g are two functions defined from set A in to set B, then

1. *Sum / difference of two functions is* **(f ± g) (x) = f(x) ± g (x)**
2. *Product of two functions is* **(f × g) (x) = f(x) × g (x)**
3. *Division of two functions is* $\left(\frac{\pi}{2},0\right)$ **(x) =** $\left(\frac{\mathbf{f}}{\mathbf{g}}\right)$ **.**

Inverse of an Element

Let us understand this with the help of an Example.

Ex.3 ***Find the inverse of y =*** $\frac{-2}{x-5}$ ***and determine whether the inverse is also a function.***

Sol. Since the variable is in the denominator, this is a rational function. Here's the algebra:

Step 1: Write the original function y = $\frac{-2}{x-5}$

Step 2: Represent x in terms of y in the equation. x = $\frac{5y-2}{y}$

Step 3: Replace the x's and y's with each other. y = $\frac{5x-2}{x}$

Thus the inverse function is y = $\frac{5x-2}{x}$ **.**

Iterative Functions

Just like the composite functions in which we perform the function g on f (x), if we perform the function f on f (x) and continue performing the resulted function on f (x), n number of times. These types of functions are called Iterative Functions.

For example: If $f^1(x) = 3x - 1$ and $f^n(x) = f^1\left(f^{n-1}(x)\right)$ for all n > 1.

Here $f^5(x) = f^1\left(f^4(x)\right) = f^1\left[f^1\left(f^3(x)\right)\right] = f^1\left[f^1\left(f^1\left(f^2(x)\right)\right)\right] = f^1\left[f^1\left[f^1\left(f^1\left(f^1(x)\right)\right)\right]\right]$

So, $f^5(1) = f^1\left[f^1\left[f^1\left(f^1\left(f^1(1)\right)\right)\right]\right]$ but f^1 (1) = 3 x 1 – 1 = 2

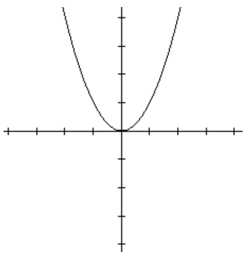
$f^5(x) = f^1 [f^1\{ f^1 (f^1\{2)\}\}] = f^1 [f^1\{ f^1 (5)\}] = f^1 [f^1(14)] = f^1(41) = 122$

Transformations of Functions

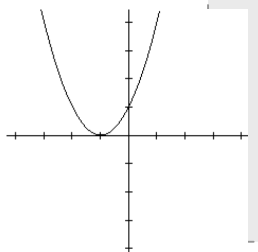
We will examine four classes of transformations:

Horizontal translation: g(x) = f(x + c).

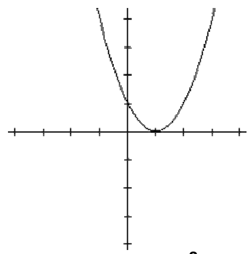
The graph is translated c units to the *left* if c > 0 and c units to the *right* if c < 0.



y = x^2



y = $(x + 1)^2$

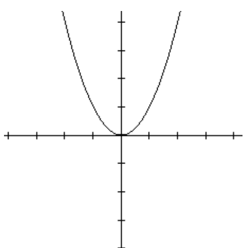


y = $(x – 1)^2$

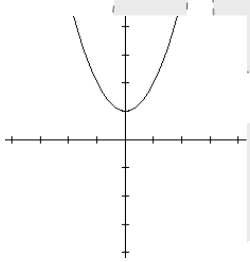
Vertical translation: g(x) = f(x) + k.

The graph is translated k units *upward* if k > 0 and k units *downward* if k < 0.

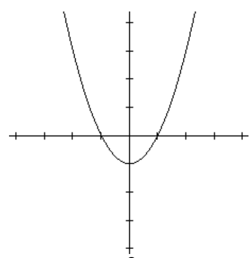
The graph is translated k units *upward* if k > 0 and k units *downward* if k < 0.



y = x^2



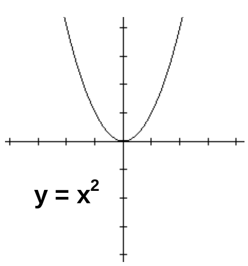
y = x^2 + 1



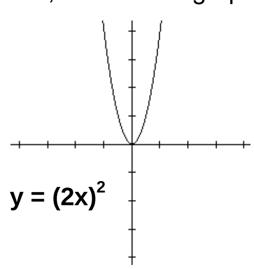
y = x^2 – 1

Change of scale: g(x) = f(Ax).

The graph is *compressed* if |A| >1 and *stretched out* if |A| > 1. In addition, if A < 0 the graph is reflected about the y-axis.



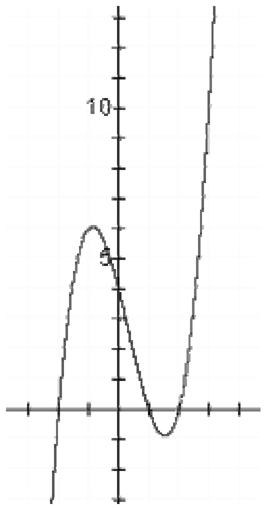
y = x^2



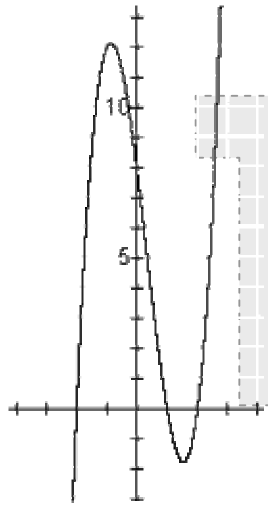
y = $(2x)^2$

Change of amplitude: g(x) = Af(x).

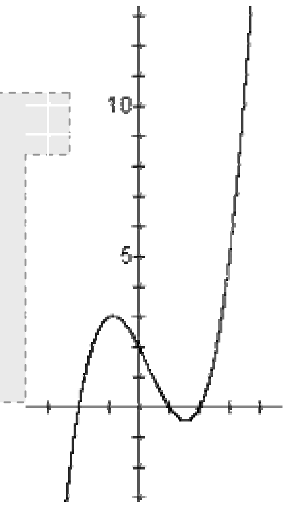
The amplitude of the graph is *increased* by a factor of A if $|A| > 1$ and *decreased* by a factor of A if $|A| < 1$. In addition, if $A < 0$ the graph is inverted.



y = f(x)



y = 2f(x)



y = 1/2f(x)

<u>Solved Examples</u>

Ex.4 **If $f(x) = 2x^2 - 3$, then f(2) = ?**

Sol. $f(2) = 2(2)^2 - 3 = \mathbf{5}$

Ex.5 $\mathbf{f(x) = \dfrac{2x^3 - k}{x + 2}}$ **if f(2) + f(3) = f(5), then K =**

Sol. $f(2) + f(3) = \dfrac{2(2)^3 - k}{2+2} + \dfrac{2(3)^3 - k}{3+2}$

$= \dfrac{16 - k}{4} + \dfrac{54 - k}{5}$

And $f(5) = \dfrac{250 - k}{7}$

$\therefore \dfrac{296 - 9k}{20} = \dfrac{250 - k}{7}$

$\Rightarrow k = \dfrac{-2928}{43}$

Ex.6 **f (n + 1) = 2f (n) + 4 & f(1) = 0. Then f (5) =**

Sol. $f(2) = 2f(1) + 4 = 0 + 4 = 4$

$f(3) = 2 \times 4 + 4 = 12$

$f(4) = 2 \times 12 + 4 = 28$

$\Rightarrow f(5) = 2 \times 28 + 4 = 60$

Ex.7 **$f(x) = x + 5$, $g(x) = x^2 - 3$. Then gofog(3) is :**

Sol gofog(3) = gof($3^2 - 3$) = gof(6) = g(6 + 5)

= g(11) = $11^2 - 3 = 118$.

Ex.8 **$f(x) = 3x^2 - 2x + 4$ then f(1) + f(2) + - - - - + f(20) =**

Sol. $= 3(\sum 20^2) - 2(\sum 20) + 4(20)$

$= 3 \times \frac{20 \times 21 \times 41}{6} - \frac{2 \times 20 \times 21}{2} + 80$

= 8610 – 420 + 80 = 8270

Ex.9 **$f(x, y) = x^y + y^x$. what can be the value of a + b, if f(a, b) = 17, a & b are integers**

Sol. $f(a, b) = a^b + b^a = 17$

2 & 3 satisfy this ($2^3 + 3^2 = 17$)

$\therefore$ a + b = 5

Ex.10 **f(x) = x + 1, g(x) = 5x -1. $f^n(x) = f\{f^{n-1}(x)\}$**

and $g^n(x)$ $g\{g^{n-1}(x)\}$. If $gof^2(k) = fog^2(k)$, find k?

Sol. gof^2 (k) = gof(k + 1) = g(k + 2) = 5k + 9

& fog^2 (k) = fog(5k – 1) = f(25k – 6) = 25 k – 5

$\therefore$ 5k + 9 = 25k – 5

$\Rightarrow k = \frac{14}{20} = \frac{7}{10}$

Ex.11 **$g(x) = \frac{1}{x}$. What is the min possible value of n such that**

g (n) – g (n + 1) < 0.02

Sol. $\frac{1}{n} - \frac{1}{n+1} < 0.02$

$\frac{1}{n(n+1)} < \frac{1}{50}$

$\Rightarrow$ n(n+1) > 50 . So, n should be 7 at least.

Derivatives

Increment

When a variable changes from one value to another, the difference between the new value and the initial value is called an increment.

If the initial value of a variable quantity says x is 9 and it changes to 9.2, the .2 is increment of x. But if x changes from value 9 to 8.9 then increment may be either positive of negative according as the variable increases or decreases during the change.

In general, remember that **derivative of a function** basically is a measure of the rate of change the function with respect to another function.

The notation for derivative is $\frac{dy}{dx}$ or f'(x).

Standard Results

Sr no	Function	Differentiation(dy/dx)
1	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
2	$y = e^x$	$\frac{dy}{dx} = e^x$
3	$y = \log x$	$\frac{dy}{dx} = \frac{1}{x}$
4	$y = a^x$	$\frac{dy}{dx} = a^x \log a$
5	$y = \sin x$	$\frac{dy}{dx} = \cos x$
6	$y = \cos x$	$\frac{dy}{dx} = -\sin x$
7	$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$

Fundamental Theorems

Some important fundamental theorems are:

Theorem 1: The differential co-efficient of a constant is zero.

For example: $\frac{d}{dx}(\pi) = 0$; $\frac{d}{dx}(c) = 0$.

$\frac{d}{dx}(K) = 0$.

Theorem 2: If u is a function of x and c is a constant then
$\frac{d}{dx}(u + c) = \frac{d}{dx}(u)$.

For example: $\frac{d}{dx}(3x^2 - 5) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5) = 6x + 0 = 6x$

And; $\frac{d}{dx}(2x - k) = \frac{d}{dx}(2x) - \frac{d}{dx}(k) = 2 - 0 = 2$.

Theorem 3: If u is a function of x and c is a constant, then
$\frac{d}{dx}(cu) = c\frac{du}{dx}$

For example: $\frac{d}{dx}3x^4 = 3.4x^{4-1} = 12x^3$.

$$\frac{d}{dx}\left(\frac{2}{5}x^{\frac{1}{2}}\right) = \frac{2}{5} \cdot \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{5\sqrt{x}} = \frac{1}{5x^{\frac{1}{2}}}$$

Theorem 4: If u, v, w, … are functions of x, then

$$\frac{d}{dx}(u \pm v \pm w + \ldots) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v) \pm \frac{d}{dx}(w) \pm \ldots$$

Simple Problems

1. Differentiate the following functions w. r. t. x

(a) $\left(x - \frac{1}{x}\right)^2$

(b) $\frac{x^4 + x^3 - 2x^2 - 4x + 6}{x^2}$

Answer: (a) $2x - \frac{2}{x^3}$ (b) $2x + 1 + \frac{4}{x^2} - \frac{12}{x^3}$

Toolkit

PRODUCT RULE

If u and v are the functions of x, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

QUOTIENT RULE

If u and v are functions of x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(\text{Denom.} \times \text{Diff. co. eff. of Num.}) - (\text{Num} \times \text{Diff. co eff. of Denom.})}{(\text{Denom.})^2}$$

Ex.1 ***Differentiate x^4 directly, taking as $\frac{x^6}{x^2}$ and $x^3 \times x$; and show that result is the same.***

Sol. Let $y = x^4$

(i) $\frac{dy}{dx} = 4x^3$

(ii) $y = x^4 = \frac{x^6}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \times \frac{d}{dx}(x^6) - x^6 \times \frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2 \times 6x^5 - x^6 \times 2x}{x^4}$$

$$= \frac{6x^7 - 2x^7}{x^4} = \frac{4x^7}{x^4} = 4x^3.$$

(iii) $y = x^4 = x^3 \times x$

$$\frac{dy}{dx} = x^3 \times \frac{d(x)}{dx} + x \times \frac{d(x^3)}{dx} = x^3 \times 1 + x \times 3x^2 = x^3 + 3x^3 = 4x^3$$

Hence the result is same in each case.

Differentiation of a function of a function

Consider $y = u^2 + 7u + 5$, and $u = x + 3$.

y is a function of u and u is a function of x.

In such a case y is said to be **function of a function.**

If y = f (u) and u = g (x), then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Applications of the Derivative

DISTANCE & SPEED

If S denotes the distance travelled by a particle and

if $S = at^3 + bt^2 + ct + d$,

Speed $= \frac{dS}{dt}$

$\Rightarrow$ Speed is the rate of change of Distance (S).

Acceleration $= \frac{d^2S}{dt^2}$

$\Rightarrow$ Acceleration is the rate of change of Speed

Ex.2 ***If $S = 3t^3 + 5t^2 - 7t + 3$ (S in cm & t in seconds), find velocity & acceleration at t = 3 sec.***

Sol. We have $\frac{dS}{dt} = 9t^2 + 10t - 7 = v$

$\Rightarrow \frac{dS}{dt} = v = 9 \times 3^2 + 10 \times 3 - 7$ (at t = 3) = 81 + 30 – 7 = 81 + 23 = 104 cm/sec.

and $a = \frac{d^2S}{dt^2} = 18\,t + 10 = 18 \times 3 + 10 = 64$ cm/sec^2 at t = 3.

MARGINAL COST

We have Marginal Cost = Derivative of total cost w.r.t. the quantity

Average cost = Total cost / Total quantity

Ex.3 ***If the average cost is denoted by $A = 3X^2 + 9X - 3$, find marginal cost when output is 2 units.***

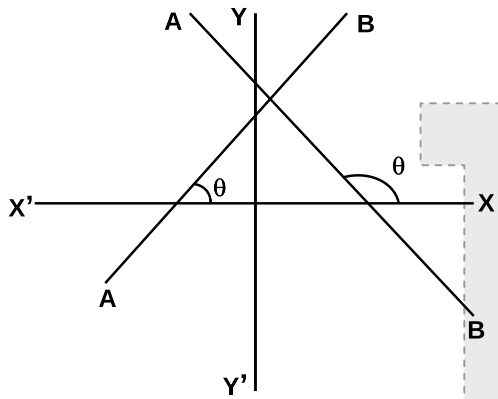
Sol. We have Total cost = X × A (X = quantity)

$\Rightarrow C = 3X^3 + 9X^2 - 3X$ & Marginal Cost $= \frac{dC}{dX} = 9X^2 + 18X - 3$

$\Rightarrow$ At X = 2, Marginal Cost = 36 + 36 – 3 = 69.

Slope of a line (m)

Slope of a line which is often denoted by **m** is equal to the tangent of the angle that it makes with x-axis.



If AB is the line making an angle θ with x-axis then, $m = \tan\theta$. As $\tan 90^o$ can not be defined hence slope of a vertical line | | to Y axis can not be defined.

Geometrically it is denoted as $\frac{dy}{dx}$.

SLOPE OF THE TANGENT

If Y = f(X) is the equation of any curve, then the slope of tangent to this curve at any point is the derivative of the equation of the curve at that point.

Ex.4 ***If $Y = X^3 + 2X^2 + 1$ is the equation of the curve, find the slope of the tangent to the curve at the point for which X = 1.***

Sol. We have $Y = 1^3 + 2 \times 1^2 + 1 = 4$ at X = 1. So the point is (1, 4).

Now slope = $\frac{dY}{dX} = 3X^2 + 4X + 0 = 7$ at X = 1.

MAXIMUM & MINIMUM VALUES OF A FUNCTION

If Y = f(X) is a function of X, then the function will assume different values for different values of X.

The value of Y = f(X) will be maximum or minimum when $\frac{dY}{dX} = 0$

Working Rule for finding maximum and minimum values of a function

Step 1: Put the function y = f (x)

Step 2: Find $\frac{dy}{dx}$ and equate it to zero.

Solve the equation $\frac{dy}{dx}$ = 0, for real values and let x = a, b, c … be its roots.

Step 3: Find $\frac{d^2y}{dx^2}$. Put x = a, b, c … in it one by one.

If $\frac{d^2y}{dx^2}$ is – ve when x = a, then f (x) is maximum at x = a and the corresponding maximum value f (x) is f (a).

If $\frac{d^2y}{dx^2}$ is + ve when x = a, then f (x) is minimum at x = a and the corresponding minimum value of f (x) is f(a).

.

Ex.5 ***Investigate for maxima & minima and thus find the maximum & minimum values of the function $F(X) = 41 + 24x - 18x^2$.***

Sol. $F(x) = 41 + 24x - 18x^2$.

$\Rightarrow \frac{dF(x)}{dx} = 24 - 36x$ and $\frac{d^2F(x)}{dx^2} = -36$.

For maximum or minimum, $\frac{dF(x)}{dx} = 0$

$\Rightarrow 24 - 36x = 0 \Rightarrow x = \frac{2}{3}$.

Now, $\left(\frac{d^2F(x)}{dx^2}\right)_{x=2/3} = -36 < 0$.

F(x) is maximum when $x = \frac{2}{3}$.

Maximum value of F (x) at $x = \frac{2}{3}$,

Which is equal to $41 + 24 \times \left(\frac{2}{3}\right) - 18\left(\frac{2}{3}\right)^2 = 49$.

Practical Problems

The method to find the maxima and minima is very much helpful in Algebra, geometry and business, and some problems regarding this can be solved easily and can be applied very usefully as illustrated below.

Ex.6 ***Divide 15 into two parts so that product of these two pars is maximum or minimum.***

Sol. Total = 15

Let one part be = x. $\therefore$ Other = 15 – x

Now y = (15 – x) x = 15x – x^2

$\therefore \frac{dy}{dx} = 15 - 2x$

Equating $\frac{dy}{dx} = 0$; we get, 15 – 2x = 0

or $x = \frac{15}{2}$ and $\frac{d^2y}{dx^2} = -2$

Hence the function has only maximum value and the point is at $x = \frac{15}{2}$

$\therefore$ Required value of $x = \frac{15}{2}$ and other part $= 15 - \frac{15}{2} = \frac{15}{2}$

So required values are $\frac{15}{2}$ and $\frac{15}{2}$ and it has no minima.

Do you know?

The area of a rectangle is maximum for a given perimeter; it must be a square.

Ex.7 ***If V denotes the volume & S the surface area of a sphere of radius r, find the rate of change of V w.r.t S when r = 2cm.***

Sol. We have to find $\frac{dV}{dS}$.

Now $V = 4/3\,\pi r^3 \Rightarrow \frac{dV}{dr} = 4/3\,\pi \times 3r^2 = 4\pi r^2$

$S = 4\pi r^2 \Rightarrow \frac{dS}{dr} = 4\pi \times 2r = 8\,\pi r$

Now $\frac{dV}{dS} = \frac{\left(\frac{dV}{dr}\right)}{\left(\frac{dS}{dr}\right)} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$

At r = 2 cm, $\frac{dV}{dS} = 1$.

Business Problems

The cost C per km. of electric cable is given by $C = \frac{120}{x} + 600\,x$, where x is the cross – section in square cms. Find the cross section for which the cost is least and also find the least cost per km.

Sol. Given $C = \frac{120}{x} + 600\,x$

$\therefore \frac{dC}{dx} = -\frac{120}{x^2} + 600$ and $\frac{d^2C}{dx^2} = \frac{240}{x^3}$

Equating $\frac{dC}{dx} = 0$; we get,

$- \frac{120}{x^2} + 600 = 0$ or $x = \pm \frac{1}{\sqrt{5}}$

– ve value is rejected and taking + ve value,

we get $x = \frac{1}{\sqrt{5}}$ [Cross section can't be – ve]

At $x = \frac{1}{\sqrt{5}}$; $\frac{d^2C}{dx^2} = 240\,(\sqrt{5}\,)^3 = 1200\sqrt{5}$ and is + ve

$\therefore$ It is point of minima.

Hence least cost per km. is $= 120 \times \sqrt{5} + 600 \times \frac{1}{\sqrt{5}}$

$= 120\sqrt{5} + 120\sqrt{5} = 240\sqrt{5}$.