Geometry – 1

(Angles, lines & properties of lines, Polygons, Triangles)

This is one of the favorite areas of CAT examination. The questions come in CAT paper are usually based on the fundamental properties of the topic and not on complex theorem. In this chapter, we will cover the fundamentals of geometry.

Angles

Based on the measurement, angles have been classified into different groups.

Complementary angles:

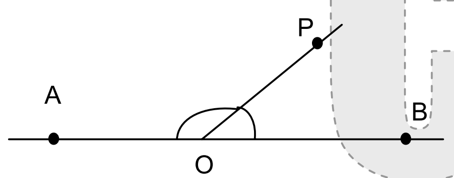
Two angles taken together are said to be complementary if the sum of measurement of the angles equal to 90^o. If $\angle A + \angle B = 90^o$ then $\angle A$ is complementary of $\angle B$ and vice – versa.

Supplementary angles:

Two angles are supplementary if sum of their measure is 180^o. If $\angle A + \angle B = 180^o$ then $\angle A$ is supplementary of $\angle B$ and vice – versa.

Linear Pair:

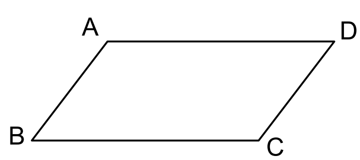
Two angle drawn on a same point and have one arm common. If sum of their measure equals to 180^o, then they are said to be liner pair of angles.



$\angle$AOP and $\angle$POB are linear pair of angles.

Adjacent angles:

Two angles are adjacent if and only if they have one common arm between them.

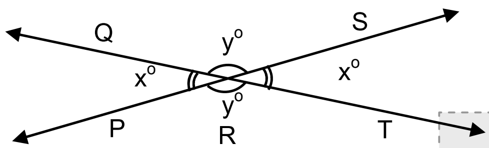


In the above figure, $\angle$ABC and $\angle$BCD are adjacent angles, since they have BC as their common arm.

Properties of Lines

A line consists of infinite dots. A line is drawn by joining any two different points on a plane. Two different lines drawn can be either parallel or intersecting depending on their nature.

If two lines intersect at a point, then they form two pairs of opposite angles (as shown in the figure), which are known as ***vertically opposite angles*** and have same measure. In the figure, $\angle PRQ$ and $\angle SRT$ are vertically opposite angles. Also $\angle QRS$ and $\angle PRT$ are vertically opposite angles.



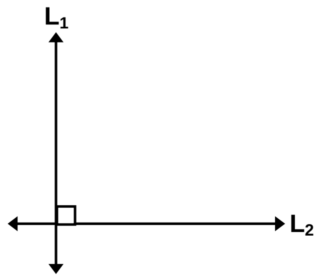
Also, $\angle x + \angle y = 180^o$ and are Linear pair angles.

Do You Know?

A right angle symbol in an angle of intersection indicates that the lines are perpendicular.

Perpendicular Lines:

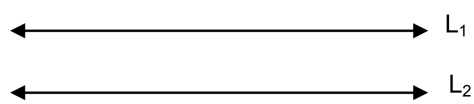
An angle that has a measure of 90^o is a right angle. If two lines intersect at right angels, the lines are perpendicular. For example:



L_1 and L_2 above are perpendicular and denoted by $L_1 \perp L_2$.

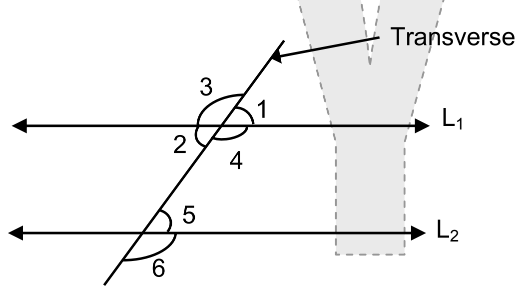
Parallel Lines:

Two lines drawn on a plane are said to be parallel if they do not intersect each other. In figure below lines, L_1 and L_2 are parallel and denoted by $L_1 || L_2$



Parallel lines and a transverse:

If a common line intersects two parallel lines L_1 and L_2, then that common line is known as transverse.



Pair of corresponding angles = ($\angle 1$ & $\angle 5$) and ($\angle 4$ & $\angle 6$)

Pair of internal alternate angles = ($\angle 2$ & $\angle 5$)

Pair of exterior alternate angles = ($\angle 3$ & $\angle 6$)

Vertically opposite angles = $\angle 3$ & $\angle 4$

For parallel lines intersected by the transversal, the pair of corresponding angles, interior alternate angles and exterior alternate angles are equal.

$\angle 1 = \angle 5$, $\angle 2 = \angle 5$, $\angle 3 = \angle 6$ and $\angle 3 = \angle 4$

Triangles

A triangle is a polygon of three sides.
Triangles are classified in two general ways: by their sides and by their angles.

Types of triangle

Based on sides, triangles have been classified into three categories.

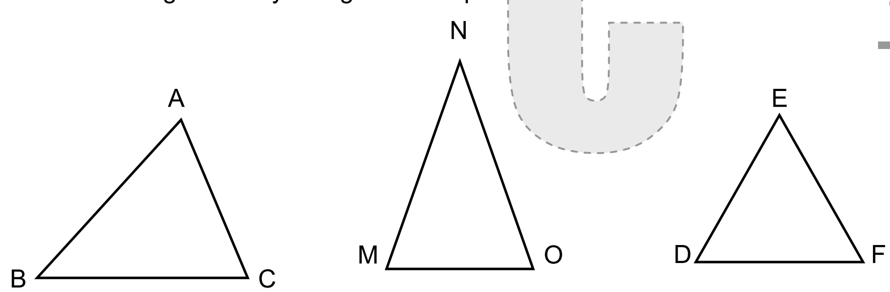
1. **Scalene triangle:**
   A triangle with three sides of different lengths is called a scalene triangle.

2. **Isosceles triangle:**
   An isosceles triangle has two equal sides. The third side is called the base. The angles that are opposite to the equal sides are also equal.

3. **Equilateral triangle:**
   An equilateral triangle has three equal sides. In this type of triangle, the angles are also equal, so it can also be called an equiangular triangle. Each angle of an equilateral triangle must measure 60^{o}, since the sum of the interior angles of any triangle must equal to 180^{o}.

TIPS

30°, 60°, 90° triangle:
This is a special case of a right triangle whose angles are 30°, 60°, 90°.
In this triangle side opposite to angle 30^{0} = Hyp/2.
Side opposite to Angle 60° = √3/2 x HYP.



scalene ***Isosceles*** ***Equilateral***

Triangles are also divided into three classes on the basis of measure of the interior angles.

Obtuse angled triangle:
When the measure of the largest angle of the triangle is greater than 90^{0} then it is an obtuse angled triangle.
In the figure ΔABC is an obtuse triangle where C is an obtuse angle

Acute angled triangle:
In which all angles are less than 90^{o} e.g.
ΔPQR is a acute triangle because largest angle is less than 90^{o}.

Do You Know?

If a, b, c denote the sides of a triangle, then
(i) Triangle is acute angled if $c^2 < a^2 + b^2$
(ii) Triangle is right angled if $c^2 = a^2 + b^2$
(iii) Triangle is obtuse angled if $c^2 > a^2 + b^2$.

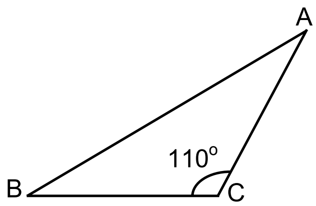
Right Angled Triangle:

A triangle whose one angle is 90^o is called a right (angled) Triangle.

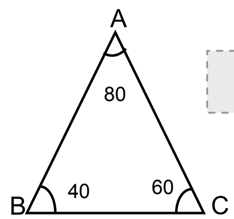
In figure, b is the hypotenuse, and a & c the legs, called base and height respectively.

In right triangle ABC we have,

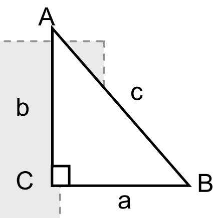
$$a^2 + b^2 = c^2$$



Obtuse angled



Acute angled



Right angled

When isosceles triangle has greatest angle as right angle then that triangle is called Right-angled isosceles triangle.

Properties of Triangle

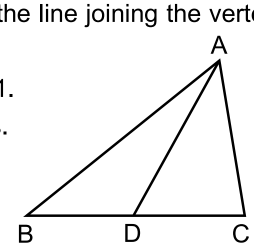
1. Sum of the three angles is 180^o.
2. Sum of exterior angles of any triangle is equal to 360^o.
3. An exterior angle is equal to the sum of the interior opposite angles.
4. The sum of the two sides is always greater than the third side.
5. The difference between any two sides is always less than the third side.
6. The side opposite to the greatest angle is the greatest side and the side opposite to the smallest angle will be the shortest side.
7. If ΔABC is a right-angled triangle then $\angle B$ is equal to 90^o unless mentioned otherwise.
8. In a right-angled triangle whose angles are 30^o, 60^o, 90^o.

   In this triangle side opposite to angle $30^0 = \frac{1}{2}$ Hypotenuse

   Side opposite to Angle $60^o = \frac{\sqrt{3}}{2}$ Hypotenuse.

9. **Centroid:**
   (a) The point of intersection of the medians of a triangle. (Median is the line joining the vertex to the mid-point of the opposite side).
   (b) The centroid divides each median from the vertex in the ratio 2 : 1.
   (c) To find the length of the median we use the theorem of Apollonius.

$$\mathbf{AB^2 + AC^2 = 2(AD^2 + BD^2)}$$



   (d) The medians will bisect the area of the triangle.
   (e) If x, y, z are the lengths of the medians through A, B, C of a triangle ABC, then
   "Four times the sum of the squares of medians is equal to three times the sum of the square of the sides of the triangle".
   $4(x^2 + y^2 + z^2) = 3(a^2 + b^2 + c^2)$.

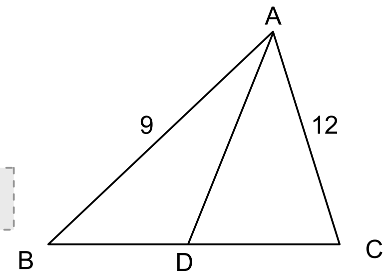
Ex.1 ***In △ABC, AB = 9, BC = 10, AC = 12. Find the length of median through A.***

Sol. In the adjacent figure AD is the required median. Using Apollonius theorem in the triangle we have,

$2AD^2 + 2(5)^2 = 81 + 144$.

$2AD^2 + 50 = 225$

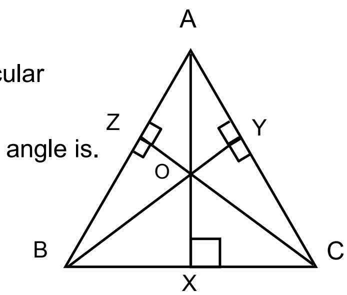
$\therefore AD = \sqrt{\frac{175}{2}} \quad = 5\sqrt{\frac{7}{2}}$



10. **Orthocentre:**

This is the point of intersection of the altitudes. (Altitude is a perpendicular drawn from a vertex of a triangle to the opposite side.)

In a right angled triangle, the orthocenter is the vertex, where the right angle is.



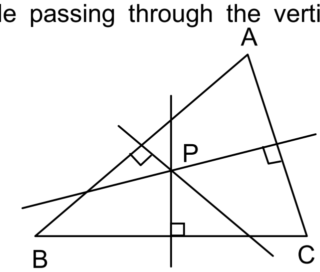
11. **Circumcentre:**

It is the point of intersection of perpendicular bisectors of the sides of the triangle

(a) The Circumcentre of a triangle is the centre of the circle passing through the vertices of a triangle.

(b) The Circumcentre is equidistant from the vertices.

(c) If a, b, c, are the sides of the triangle, Δ is the area, then abc = 4R Δ where R is the radius of the circum-circle.

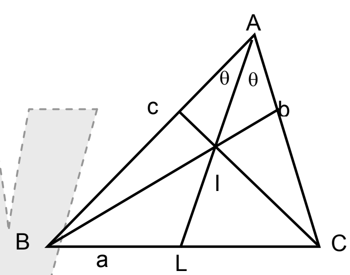


12. **Incentre:**

This is the point of intersection of the internal bisectors of the angles of a triangle.

(a) $\frac{BL}{LC} = \frac{AB}{AC}$

(b) $\frac{AI}{IL} = \frac{b+c}{a}$



(c) Δ = rs if r is the radius of incircle, where s = semi-perimeter = $\frac{a+b+c}{2}$

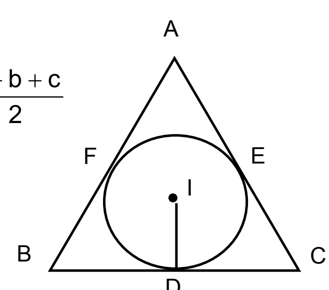
and Δ is the area of the triangle.

(d) BF = BD = s – b where 2s = a + b + c,

CE = CD = s – c

AF = AD = s – a

(e) The angle between the internal bisector and the external bisector is 90°.



Ex.2 ***The sides of the triangle are 6 cm, 8 cm, ad 10 cm. Find the area, Inradius and Circumradius of the triangle.***

Sol. s = $\frac{6+8+10}{2} = 12$

Area = Δ = $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{12\cdot 6\cdot 4\cdot 2} = 24\,\text{sq.cm.}$

$4R\Delta = abc$; $4 \times R \times 24 = 6 \times 8 \times 10$

$\therefore R = 5$ cm

Now, r = $\frac{\Delta}{s} = \frac{24}{12} = 2$ cm.

Equilateral triangle

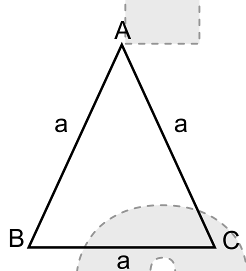
In an equilateral triangle all the sides are equal and all the angles are equal.

(a) Altitude = $\frac{\sqrt{3}}{2} \times \text{side} = \frac{\sqrt{3}}{2} a$

(b) Area = $\frac{\sqrt{3}}{4} \times (\text{side})^2 = \frac{\sqrt{3}}{4} a^2$

(c) Inradius = $\frac{1}{3} \times$ Altitude

(d) Circumradius = $\frac{2}{3} \times$ Altitude



TIPS

(i) The equilateral Δ has maximum area for given the perimeter,

(ii) Of all the triangles that can be inscribed in a given circle, an equilateral triangle has maximum area.

Congruency

Two or more figures can be said congruent if and only if they all have same size and shape. If we talk about plane figures then they are congruent if their corresponding sides and angles are equal to the corresponding sides and angles of the other figure. E.g.:- Two triangles ABC and DEF are said to the congruent, if they are equal in all respects (equal in shape and size).

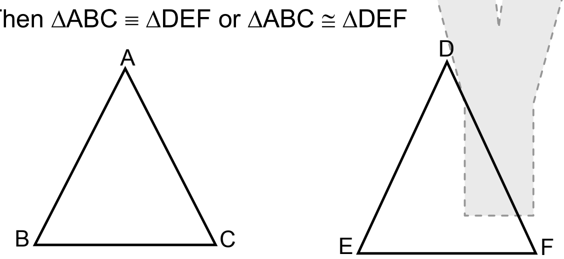
The notation for congruency is $\cong$ or $\equiv$

Congruent triangles

If $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

AB = DE, BC = EF; AC = DF

Then $\Delta ABC \equiv \Delta DEF$ or $\Delta ABC \cong \Delta DEF$



The tests for congruency

(a) ***SAS Test:*** Two sides and the included angle of the first triangle are respectively equal to the two sides and included angle of the second triangle.

(b) ***SSS Test:*** Three sides of one triangle are respectively equal to the three sides of the other triangle.

(c) ***ASA Test:*** Two angles and one side of one triangle are respectively equal to the two angles and one side of the other triangle.

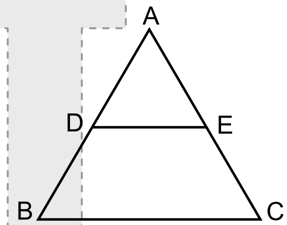
(d) ***RHS Test:*** The hypotenuse and one side of a right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle.

Mid-point Theorem

A line joining the mid points of any two sides of a triangle must be parallel to the third side and equal to half of that (third side).

In the adjacent triangle ABC, if D and E are the respective mid-points of sides AB & AC, then

DE II BC and DE = $\frac{1}{2}$BC



Similar triangles

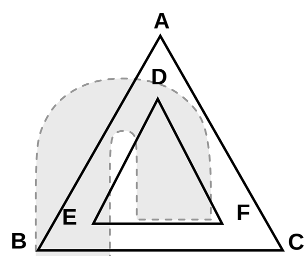
Two figures are said to be similar, if they have the same shape but not necessarily the same size. If two triangles are similar, the corresponding angles are equal and the corresponding sides are proportional.

In the figure

$\Delta ABC \sim \Delta DEF$ then,

$\angle A = \angle D$, $\angle B = \angle E$ & $\angle C = \angle F$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$



Do You Know?

1. All Congruent triangles are similar but similar triangles need not be congruent.

2. All equilateral triangles are similar.

Test for similarity of triangles

(a) **AAA Similarity Test:** Three angles of one triangle are respectively equal to the three corresponding angles of the other triangle.

(b) **SAS Similarity Test:** The ratio of two corresponding sides is equal and the angles containing the sides are equal.

(c) **SSS Similarity Test:** The ratio of all the three corresponding side of the two triangles are equal.

Basic proportionality Theorem

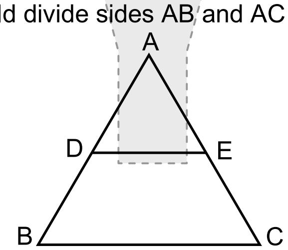
In a triangle if a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the other two sides proportionally.

If in ΔABC, DE is drawn parallel to BC, it would divide sides AB and AC proportionally i.e. $\frac{AD}{DB} = \frac{AE}{EC}$

We can also use the following results:

(i) $\frac{AB}{AD} = \frac{AC}{AE}$

(ii) $\frac{AD}{DE} = \frac{AB}{BC}$

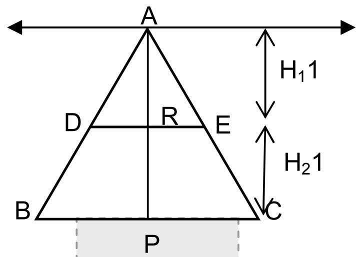


Important Result

If in ΔABC DE II BC, and a line is drawn passing thru A and parallel to BC.

Then we will have

$$\frac{AD}{BD}=\frac{AR}{PR}=\frac{AE}{CE}=\frac{H_1}{H_2}=\frac{DE}{BC}$$



These results are such because of DE being parallel to BC.

Areas of similar triangles

The ratios of the areas of two similar triangles are equal to the ratio of the square of their corresponding sides i.e. If $\Delta ABC \sim \Delta DEF$ then

$$\frac{ar(\Delta ABC)}{ar(\Delta DEF)}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{CA^2}{FD^2}$$

The ratio of the areas of two similar triangles is also equal to

(a) Ratio of the square of their corresponding medians.

(b) Ratio of the square of their corresponding Altitudes.

(c) Ratio of the square of their corresponding angle bisectors.

Properties

If two triangles are similar, the following properties are true:

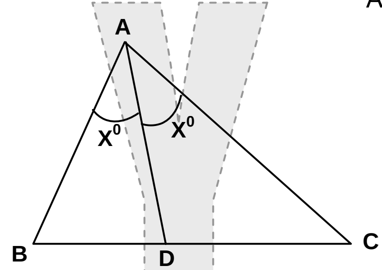
(a) The ratio of the medians is equal to the ratio of the corresponding sides.

(b) The ratio of the altitudes is equal to the ratio of the corresponding sides.

(c) The ratio of the internal bisectors is equal to the ratio of the corresponding sides.

Angle Bisector Theorem

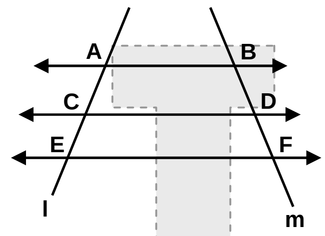
The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle. i.e. In a Δ ABC in which AD is the bisector of $\angle$A, then $\frac{BA}{AC}=\frac{BD}{DC}$.



Intercept Theorem

Intercepts made by two transversals (cutting lines) on three or more parallel lines are proportional. In the figure, lines l and m are transversals to three parallel lines AB, CD, EF. Then, the intercepts (portions of lengths between two parallel lines) made, AC, BD & CE, DF, are respectively proportional.

$$\frac{AC}{BD} = \frac{CE}{DF}$$



Pythagoras Theorem

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides i.e. in a right angled triangle ABC, right angled at C,

$$AB^2 = AC^2 + BC^2$$

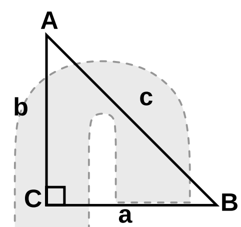
Or $a^2 + b^2 = c^2$

Do You Know?

. If you multiply the Pythagorean triplets by constant, then resultant will also be Pythagorean triplets e.g. (6, 8, 10), (18, 24, 30) etc.

Pythagorean Triplets:

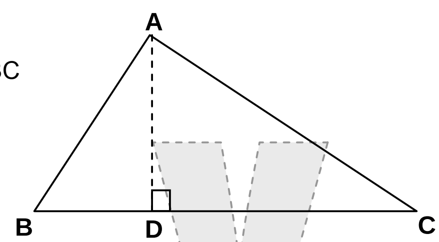
Pythagorean triplets are sets of 3 Integers which can be taken as the three sides of a right-angled triangle. Few Pythagorean triplets are (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41) etc.



Acute angled theorem

In an acute angle triangle ABC, AD is the altitude on BC from vertex A, and ∠ABC is the greatest angle among all the three angles. Then

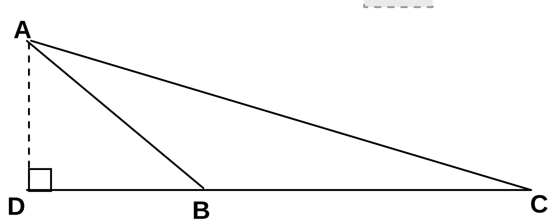
$$AC^2 = AB^2 + BC^2 - 2BD \times BC$$



Obtuse angled theorem

In an obtuse angle triangle ABC, AD is the altitude on CB produced from vertex A, and ∠ABC is the greatest angle among all the three angles. Then

$$AC^2 = AB^2 + BC^2 + 2BD \times BC$$



Apollonius' theorem

This theorem is the combination of above two theorems and gives the length of the median. If in ΔABC, AD is the median, meeting side BC at D. Then

$$AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$$