Inequalities

(Solving linear inequations, Solution sets, Graphical Solution)

Introduction

The expressions like $3x + 1 = x - 3$ and $x^2 - 3x = 0$ are ***equations***. But which type of expressions is called **"*inequations*"**. When a statement written is neither true nor false until and unless it is replaced by some numeral value, is called an open sentence. E.g. $7x + 5 > 19$ is such that no conclusion can be made from this unless x is replaced by some numeral values.

Now we can define inequalities as "a sentence which says that one thing is not equal to another. The two sides are joined together by one of the following:

- **>** (greater than);
- **<** (less than);
- ≥ (greater than or equal to)
- ≤ (less than or equal to)

Example: $3x + 1 > x - 3$ or $x^2 - 3x \leq 0$

As is the case with equations, they are ordered by degree and by the number of unknowns.

In the above two examples, the first will be a linear inequality with one unknown and the second will be quadratic inequality with one unknown.

Solving Inequation

On the basis of the laws of inequality, we have the following working rules.

1. ***Rule of Addition and Subtraction:*** Adding or subtracting a fixed number to each side of an inequality produces an equivalent inequality.
Example: Adding 2 to each side of the inequality $x - 2 \leq 1$ is equivalent to $x \leq 3$.

2. ***Rule of multiplication/division by a positive number***: All terms on both sides of an inequality can be multiplied or divided by a positive number.

3. ***Rule of multiplication/division by a negative number:*** If all terms on both sides of an inequality are multiplied or divided by a negative number, the sign of the inequality will be reversed.

Example: If $6x - 4 < -8x + 6$ then $\frac{6x}{-2} - \frac{4}{-2} > \frac{-8x}{-2} + \frac{6}{-2}$ **or** $-3x + 2 > 4x - 3$

Using the above facts we take the following steps to solve linear inequation:

$$ax + b > cx + d.$$

Step 1: Make all terms containing the variable (unknown) x positive and on one side of the inequation, and the constants on the other side by using rule 1.

Step 2: Put the inequation in the form $px > q$ or $px < q$

Step 3: Divide both sides by positive p, using rule 3.

Graphing Solutions of Inequalities

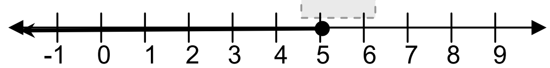
Representation of the real number set R and its subsets:

The solution of a linear inequation in one variable consists of many real numbers but may not all real numbers. As such the set of solutions is always a subset of real numbers.

Emphasize the fact that solving an inequality means describing a set not just finding a number. This set is the solution set of the problem. To visualize the solution set, draw it on the number line. To do this, draw the number line and shade the part of the number line corresponding to the solution set. The solution set to a single inequality will be a half line that may or may not include the endpoint. When the endpoint is included, draw a solid dot for the endpoint. When the endpoint is not included, draw a circle.

Ex.1 ***Represent the solution set of $x \leq 5$ on a number line.***

Sol. The solution set of $x \leq 5$ is represented by the set of point to the left of 5 on the number line. It does include the endpoint, since $x = 5$ is included.



Solution Sets

The solution of any inequation is always expressed in a solution set. These solution the basis of requirement of the elements contained in them can be classified as below:

a. **Ranges where the ends are excluded:** If the value of x is denoted as (1, 2) it means $1 < x < 2$ i.e. x is greater than 1 but smaller than 2.

b. **Ranges where the ends are included**

[2, 5] means $2 \leq x \leq 5$

c. **Mixed Ranges**

(3, 21] means $3 < x \leq 21$

d. If solution set = {1, 2} means x = 1 and 2 only.

Ex.2 ***Solve for x, $-2x > 4$***

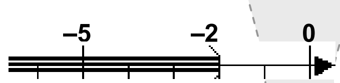
Sol. $-2x > 4$

Divide both sides by – 2,

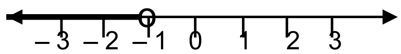
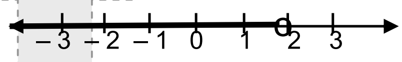
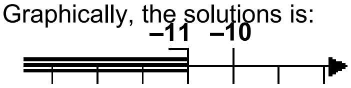
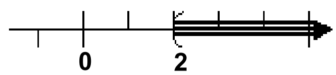
$$\frac{-2x}{-2} < \frac{4}{-2}$$

$\Rightarrow x < -2$

Graphically, the solution is:



Some more examples:

x + 3 < 2 Adding -3 we get $\underline{-3 \qquad -3}$ $x < -1$ Graphically, the solution is:  	**2 – x > 0** Adding x we get $\underline{x \qquad x}$ $2 > x$ or x < 2 Graphically, the solution is: 
4x + 6 ≤ 3x – 5 $\underline{-3x \qquad -3x}$ $x + 6 \le -5$ $\underline{-6 \quad -6}$ $x \le -11$ Graphically, the solutions is:  	**2x > 4** x > 2 (Dividing by 2) Graphically, the solution is: 

Ex.3 ***P is the solution set of $\frac{1}{x} > \frac{3}{4}$ and R is the solution set of : $x\left(1-\frac{1}{x}\right) \ge 5(x-1)$. Where x $\in$ W, find the set P $\cap$ R.***

Sol. For finding the solution set of P solve the inequation.

$$\frac{1}{x} > \frac{3}{4} \Rightarrow x < \frac{4}{3}$$

i.e. x < 1.33, but x $\in$ W.

So solution set of P = {1} since for x = 0 the inequation is not defined

Now for solution set of R

$x\left(1-\frac{1}{x}\right) \ge 5(x-1) \Rightarrow x - 1 \ge 5x - 5$ when $x \ne 0$.

Solving we have, $x \le 1$ but $x \ne 0$. Hence solution set R = {1}

P $\cap$ R = {1}.

Properties of inequalities

(i) If a > b and b > c, then a > c.

(ii) If a > b and c is any real number, then a + c > b + c and a – c > b – c.

(iii) If $a_1 > b_1$, $a_2 > b_2$, ….$a_n > b_n$, then $a_1 + a_2 + \ldots\ldots a_n > b_1 + b_2 + \ldots.b_n$.

(iv) If $a_1 > b_1$, $a_2 > b_2$, ……..$a_n > b_n$ and the a's and the b's are all positive real numbers. Then $a_1 . a_2 \ldots a_n > b_1 . b_2 \ldots b_n$.

(v) If a > b > 0, then $\frac{1}{a} < \frac{1}{b}$.

(vi) If a > b > 0 and n is a positive integer, then $a^n > b^n$, $a^{-n} < b^{-n}$ and $a^{1/n} > b^{1/n}$ provided in the last inequality only real positive values of the n^{th} roots are taken into account.

> ***Do you know?***
> If $a_1 > b_1$ and $a_2 > b_2$, then it does not necessarily follow that $a_1 - a_2 > b_1 - b_2$.

(vii) If $x > y > 0$, then $\log_a x > \log_a y$ if $a > 1$and $\log_a x < \log_a y$ if $0 < a < 1$.

(viii) If $x > y > 0$, then $a^x > a^y$ if $a > 1$, and $a^x < a^y$ if $0 < a < 1$.

(ix) If $a > b > 0$ and $x > 0$, then $a^x > b^x$.

(x) The Arithmetic mean of n positive numbers is always greater than their Geometric means.

i.e. if $a_1, a_2. \ldots. a_n$ are all positive then $\frac{a_1 + a_2 ... + a_n}{n} > (a_1 a_2 \ldots a_n)^{1/n}$.

CHALLENGER

List one of the solution set of $\frac{1}{8} < \frac{m}{n} < \frac{1}{7}$, where $m, n \in Z$

(xi) If the sum of n positive numbers is a constant, then their product is maximum when they are all equal; if their product is a constant their sum is minimum when they are equal.

i.e. If $a_1 + a_2 + \ldots + a_n$ = constant k, where a's are all positive, then a_1 .

$a_2 \ldots a_n$ is maximum when $a_1 = a_2 = a_3 = \ldots a_n = \frac{k}{n}$.

Also, if $a_1 a_2 a_3 \ldots a_n$ = constant k, where the a's are all positive, then $a_1 + a_2 + \ldots + a_n$ is Minimum when $a_1 = a_2 = \ldots a_n = k^{1/n}$.

(x) $a^2 + b^2 + c^2 \geq ab + bc + ca$

(xi) $(n!)^2 > n^n$, for all $n > 2$

(xii) For integer, $2 \leq \left(1 + \frac{1}{2}\right)^n \leq 3$

(xiii) $a^2b + b^2c + c^2a \geq 3abc$

(xiv) $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$

(xv) $a^4 + b^4 + c^4 + d^4 \geq 4abcd$

Ex.4 What values for X satisfy the inequality $x^2 - 10x + 21 < 0$

(1) $x < 3$ (2) $x > 7$ (3) $3 < x < 7$ (4) No solution.

Sol. $x^2 - 10x + 21 < 0$ means $(x - 3)(x - 7) < 0$. So $x - 3 < 0$ and $x - 7 > 0$ or $x - 3 > 0$ and $x - 7 < 0$. By drawing on the number line, the intersecting part will be $3 < x < 7$.

Answer: (3)

(or)

Go with the options.

It is very simple to solve the inequality questions from the options.

Take option (1), since it is x<3, take any value for x, which is less than 3. If the inequality does not satisfy, then we can eliminate this option. Some times two options satisfy the inequality. Then we have to pick the option which has maximum range.

Here x = 2 does not satisfy the inequality, hence option (1) is eliminated.

Similarly x = 8 does not satisfy. So option (2) also can be eliminated.

Since the inequality is satisfied for x = 4, 5,… Option (3) is the required answer.

Ex.5 If $x^3 - 3x^2 - 6x + 8 > 0$; then X can be best described by,

(1) $x < 4$ (2) $-2 < x < 1$ (3) $x > 4$ (4) both (2) and (3)

Sol. It is better to solve this with the help of options

x = 3 does not satisfy the inequality, so option (1) is eliminated.

The inequality is satisfied by all the values of option (2) and the values of option (3) also satisfy this. So the required. **Answer: (4)**

Solutions of Quadratic Inequalities

Solving linear inequalities, such as "$x + 3 > 0$", was pretty straightforward, as long as you remembered to flip the inequality sign whenever you multiplied or divided throughout by a negative number (as you would when solving something like "$-2x < 4$").

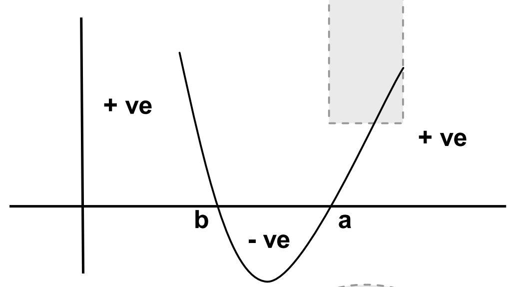
We can follow the same method of finding intercepts and using graphs to solve inequalities containing quadratics. When we have an inequality with "x^2" as the highest-degree term, it is called a "quadratic inequality".

Let us consider a quadratic equation having two roots a and b, such that a > b. then

(i) $(x - a)(x - b) > 0$

(ii) $(x - a)(x - b) < 0$ are the possible inequalities.

If we plot the graph of the quadratic equation with roots a and b, we get



So from the graph the solution of the above two equations are

(i) Either $x < b$ or $x > a$

(ii) But on the other end the value $b < x < a$ is the solution of equation 2.

Ex.6 $-x^2 + 4 < 0.$

Sol. $-x^2 + 4 < 0.$

$4 - x^2 < 0.$

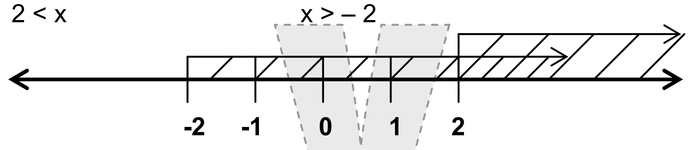
$(2 - x)(2 + x) < 0.$

The inequality will be less than zero if we have any one among the two is negative and other positive

CASE 1. $2 - x < 0$ and $2 + x > 0$

$2 - x + x < 0 + x$ $\quad$ $2 + x - 2 > 0 - 2$

$2 < x$ $\quad$ $x > -2$



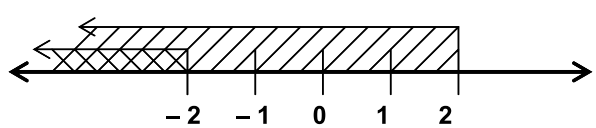
From the number line we have

$x > 2$ ……..(1)

CASE 2 $2 - x > 0$ and $2 + x < 0$

$2 - x + x > 0 + x$ $\quad$ $2 + x - 2 < 0 - 2$

$2 > x$ and $x < -2$



So the solution set **$x < -2$**

Solution Set = {**$x > 2$ or $x < -2$**}

Higher Degree Inequations

The polynomial inequations of degree 3 or more are polynomials of higher degree. Let us consider an inequation of degree 3.

$(x + 1)(x – 2)(x – 4) > 0$ ………….. (1)

Well this is the inequation of as the multiplication of three numbers whose result should be positive i.e. p x q x r > 0 where p, q & r are real numbers. The product of three real numbers should be positive if all of them are positive or at least two of them positive.

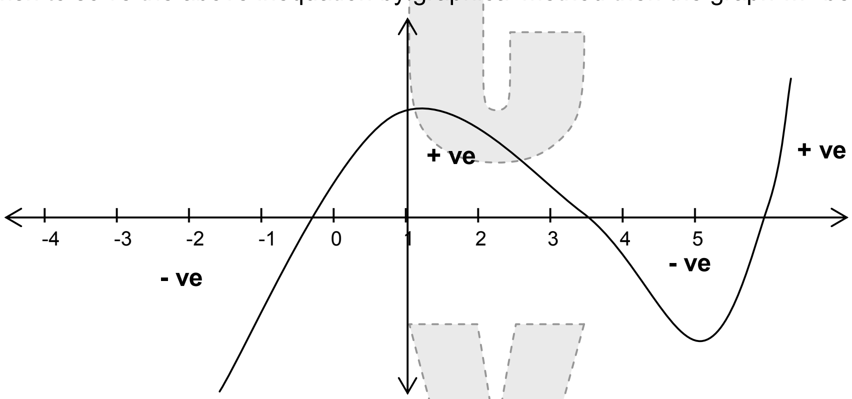
In the given equation we have three points where the inequation change its nature. These points are x = (– 1, 2, 4). The nature of inequation can be explained below in the table –

Value of x	**(x + 1)**	**(x – 2)**	**(x – 4)**	**(x + 1) (x – 2) (x – 4)**
x < -1	– ve	–ve	–ve	(x + 1) (x – 2) (x – 4) < 0
–1< x < 2	+ ve	–ve	–ve	(x + 1) (x – 2) (x – 4) > 0
2< x < 4	+ ve	+ ve	–ve	(x + 1) (x – 2) (x – 4) < 0
x > 4	+ ve	+ ve	+ ve	(x + 1) (x – 2) (x – 4) > 0

So from the table we have the solution set of the inequation.

Solution Set = **$– 1 < x < 2$** or **$x > 4$**

If we wish to solve the above inequation by graphical method then the graph will be as follows



Ex.7 ***Solve the inequation graphically and give the general solution of both the equations.***

i. ***$(X – a)(X – b)(X – c)(X – d) > 0$***

ii. ***$(X – a)(X – b)(X – c)(X – d) < 0$,***

Where a, b, c & d are real numbers and $a < b < c < d$

Absolute value

The absolute value |x|, read as the **modulus** of x of a real number x is defined as

$|x| = x$ if $x \geq 0$ and $|x| = – x$, if $x < 0$.

Thus $|x| \geq 0$; $|x| = |– x|$

Modulus and inequality

General solution of the inequation containing modulus:

(a) If $|x| = c$ $\Rightarrow$ $\mathbf{x = \{-c, c\}}$

(b) If $|x| > c$ $\Rightarrow$ $\mathbf{x > c}$ or $\mathbf{x < -c}$

(c) If $|x| < c$ $\Rightarrow$ $\mathbf{-c < x < c}$.

Ex.8 ***What is the solution set of equation*** $x^2 + 5|x| + 6 > 0$ ***?***

Sol. **Case 1:** If $x \geq 0$, then $|x| = x$

So we have, $x^2 + 5x + 6 > 0$

$\Rightarrow (x + 2)(x + 3) > 0$

When both positive we get, $x > -2$ but $x \geq 0$

Hence solution set = $\{x \mid x \geq 0 \text{ and } x \in R\}$ …………(1)

When both negatives then we have, $x < -3$ but x is positive, So no solution.

Case 2: If $x < 0$, then $|x| = -x$

We have, $x^2 - 5x + 6 > 0$

$\Rightarrow (x - 2)(x - 3) > 0$

Solving the inequality we will have $x > 3$ or $x < 2$ but here $x < 0$

So solution set = $\{x \mid x < 0 \ \& \ x \in R\}$ ……………(2)

From equation 1 and 2 we have $x \in R$ is the solution of the given equation.

Toolkit

(i)	$|x - a| = x - a$ if $x > a$ and $|x - a| = a - x$, if $x < a$
(ii)	If $b > 0$, then $|x - a| < b \Rightarrow -b < x - a < b$ or $a - b < x < a + b$
	In particular, if $b > 0$, then $|x| < b \Rightarrow -b < x < b$
(iii)	$|x - a| > b \Rightarrow x - a > b$ or $a - x > b$
	$\Rightarrow x > a + b$ or $x < a - b$
(iv)	$|x - a| = b \Rightarrow x - a = b$ or $a - x = b \Rightarrow x = a + b$ or $x = a - b$
(v)	$|a + b| \leq |a| + |b|$, i.e. (sum) $\leq$ sum of moduli
(vi)	$|a - b| \geq |a| - |b|$
(vii)	$|a. b| = |a| . |b|$, mod (product) = product of moduli

Ex.9 ***Solve*** $|2x + 3| < 6$.

Sol. Since this is a "less than" absolute value inequality, the first step is to clear the absolute value according to the pattern:

$|2x + 3| < 6$

$-6 < 2x + 3 < 6$ [this is the pattern for "less than"]

$-6 - 3 < 2x + 3 - 3 < 6 - 3$

$-9 < 2x < 3$

$^{-9}/_2 < x < {}^3/_2$

Then the solution to $|2x + 3| < 6$ is the interval $-\dfrac{9}{2} < x < \dfrac{3}{2}$.

Best approach to solve the inequality questions is to approach the questions by options.

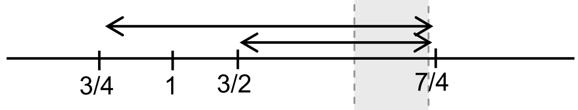
Ex.10 ***Solution of the inequality: $|x + 2| > |3x - 5|$ is***

(1) $\frac{3}{2} < x < \frac{7}{2}$ *(2)* $\frac{3}{4} < x < \frac{7}{2}$ *(3)* $\frac{1}{2} < x < 1$

(4) $\frac{3}{4} > x > \frac{7}{2}$.   *(5)* $-\frac{3}{2} < x < \frac{7}{2}$

Sol. Pick any option & check any value of x satisfying the inequality.

Let us take x = 2 which lies in option (1) & (2) & (5), If x = 2 satisfies the given equation then one out of (1) or (2) or (5) is the answer otherwise one out of (3) or (4) is the answer.



Put x = 2 in $|x + 2| > |3x - 5|$ we get

4 > 1 which is true so choice (3) & (4) can be eliminated

Put x = 1 we get

3 > 2 which is true and hence (1) is eliminated, because it doesn't has x = 1.

Now take x = – 1, we get 1 > 8, which is wrong, so option (5) is eliminated.

So answer is (2).