Finding Remainders of a product (derivative of remainder theorem)

If 'a_1'is divided by 'n', the remainder is 'r_1' and if 'a_2' is divided by n, the remainder is r_2. Then **if a_1+a_2 is divided by n, the remainder will be $r_1 + r_2$**

If $a_1 - a_2$ is divided by n, the remainder will be $r_1 - r_2$

If $a_1 \times a_2$ is divided by n, the remainder will be $r_1 \times r_2$

Ex. If 21 is divided by 5, the remainder is 1 and if 12 is divided by 5, the remainder is 2.
Then if (21 + 12 = 33) is divided by 5, the remainder will be 3 (1 + 2).
If 9(21 – 12) is divided by 5, the remainder will be 1 – 2 = – 1.
But if the divisor is 5, – 1 is nothing but 4. $9 = 5 \times 1 + 4$.
So, if 9 is divided by 5, the remainder is 4 and 9 can be written as $9 = 5 \times 2 - 1$.
So here – 1 is the remainder. So – 1 is equivalent to 4 if the divisor is 5. Similarly – 2 is equivalent to 3.
If 252(21 × 12) is divisible by 5, the remainder will be (1 × 2 = 2).

If two numbers 'a_1' and 'a_2' are exactly divisible by n. Then their sum, difference and product is also exactly divisible by n.

i.e., **If 'a_1' and 'a_2' are divisible by n, then**

$a_1 + a_2$ is also divisible by n

$a_1 - a_2$ is also divisible by n

and If $a_1 \times a_2$ is also divisible by n.

Ex.1 12 is divisible by 3 and 21 is divisible by 3.

Sol. So, 12 + 21 = 33, 12 – 21 = – 9 and 12 × 21 = 252 all are divisible by 3.

Finding Remainders of powers with the help of Remainder theorem:

Ex.2 ***What is the remainder if 7^{25} is divided by 6?***

Sol. If 7 is divided by 6, the remainder is 1. So if 7^{25} is divided by 6, the remainder is 1^{25} (because $7^{25} = 7 \times 7 \times 7\ldots$ 25 times. So remainder = $1 \times 1 \times 1\ldots$. 25 times = 1^{25}).

Ex.3 What is the remainder, if 3^{63} is divided by 14.

Sol. If 3^3 is divided by 14, the remainder is – 1. So 3^{63} can be written as $(3^3)^{21}$.
So the remainder is $(-1)^{21} = -1$. If the divisor is 14, the remainder – 1 means 13. (14 – 1 = 13)

By pattern method

Ex.4 ***Find remainder when 4^{33} is divided by 7.***

Sol. If 4^1 is divided by 7, the remainder is 4. ($4^1 = 4 = 7 \times 0 + 4$)
If 4^2 is divided by 7, the remainder is 2 ($4^2 = 16 = 7 \times 2 + 2$)
If 4^3 is divided by 7, the remainder is 1 ($4^3 = 4^2 \times 4$, So the Remainder = $4 \times 2 = 8 = 1$)
If 4^4 is divided by 7, the remainder is 4 ($4^4 = 4^3 \times 4$, so the Remainder = $1 \times 4 = 4$)

$4 \rightarrow$	4
$4^2 \rightarrow$	2
$4^3 \rightarrow$	1
$4^4 \rightarrow$	4
$4^5 \rightarrow$	2

The remainders of the powers of 4 repeats after every 3^{rd} power.
So, as in the case of finding the last digit, since the remainders are repeating after every 3^{rd} power, the remainder of 4^{33} is equal to the remainder of 4^3 (since 33 is exact multiple of 3) = 1. **(OR)**
If 4^3 is divided by 7, the remainder is 1. So $4^{33} = (4^3)^{11}$ is divided by 7, the remainder is $1^{11} = 1$.

Application of Binomial Theorem in Finding Remainders

The binomial expansion of any expression of the form
$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} \times b^1 + {}^nC_2 \times a^{n-2} \times b^2 + \ldots\ldots + {}^nC_{n-1} \times a^1 \times b^{n-1} + {}^nC_n \times b^n$
Where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ….. are all called the ***binomial coefficients***.

In general, ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

There are some fundamental conclusions that are helpful if remembered, i.e.

a. There are (n + 1) terms.
b. The first term of the expansion has only a.
c. The last term of the expansion has only b.
d. All the other (n – 1) terms contain both a and b.
e. If $(a+b)^n$ is divided by a then the remainder will be b^n such that $b^n < a$.

Ex.5 ***What is the remainder if 7^{25} is divided by 6?***

Sol. $(7)^{25}$ can be written $(6 + 1)^{25}$. So, in the binomial expansion, all the first 25 terms will have 6 in it. The 26^{th} term is $(1)^{25}$. Hence, the expansion can be written 6x + 1.
6x denotes the sum of all the first 25 terms.
Since each of them is divisible by 6, their sum is also divisible by 6, and therefore, can be written 6x, where x is any natural number.
So, 6x + 1 when divided by 6 leaves the remainder 1. **(OR)**
When 7 divided by 6, the remainder is 1. So when 7^{25} is divided by 6, the remainder will be $1^{25} = 1$.

Wilson's Theorem

If n is a prime number, (n – 1)! + 1 is divisible by n.
Let take n = 5
Then (n – 1)! + 1 = 4! + 1 = 24 + 1 = 25 which is divisible by 5.
Similarly if n = 7
(n – 1)! + 1 = 6! + 1 = 720 + 1 = 721 which is divisible by 7.

Corollary

If (2p + 1) is a prime number $(p!)^2 + (-1)^p$ is divisible by 2p + 1.
e.g If p = 3**,** 2p + 1 = 7 is a prime number
$(p!)^2 + (-1)^p = (3!)^2 + (-1)^3 = 36 - 1 = 35$ is divisible by (2p + 1) = 7.

Property

If "a" is natural number and P is prime number then $(a^p - a)$ is divisible by P.
e.g. If 2^{31} is divided by 31 what is the remainder?

$\dfrac{2^{31}}{31} = \dfrac{2^{31} - 2 + 2}{31}$ So remainder = 2

Fermat's Theorem

If p is a prime number and N is prime to p, then $N^{p-1} - 1$ is a multiple of p.

Corollary

Since p is prime, p – 1 is an even number except when p = 2.

Therefore $(N^{\frac{p-1}{2}} + 1)(N^{\frac{p-1}{2}} - 1) = M(p)$.

Hence either $N^{\frac{p-1}{2}} + 1$ or $N^{\frac{p-1}{2}} - 1$ is a multiple of p, that is $N^{\frac{p-1}{2}} = Kp \pm 1$, where, K is some positive integer.

Base Rule and Conversion

This system utilizes only two digits namely 0 & 1 i.e. the base of a binary number system is two.

e.g. 1101_2 is a binary number, to find the decimal value of the binary number, powers of 2 are used as weights in a binary system and is as follows:

$1 \times 2^3 =$ 8

$1 \times 2^2 =$ 4

$0 \times 2^1 =$ 0

$1 \times 2^0 =$ 1

Thus, the decimal value of 1101_2 is $1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13$.

Conversion from decimal to other bases

We will study only four types of Base systems,

1. Binary system (0, 1)
2. Octal system (0, 1, 2, 3, 4, 5, 6, 7).
3. Decimal system (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
4. Hexa-decimal system (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C D, E, F) where A = 10, B = 11 … F = 15.

Let us understand the procedure with the help of an example

Ex.6 ***Convert*** $\boldsymbol{357_{10}}$ ***to the corresponding binary number.***

Sol. To do this conversion, you need to divide repeatedly by 2, keeping track of the remainders as you go. Watch below:

As you can see, after dividing repeatedly by 2, we end up with these remainders:

2	357	1
2	178	0
2	89	1
2	44	0
2	22	0
2	11	1
2	5	1
2	2	0
	1	

These remainders tell us what the binary number is! Read the numbers outside the division block, starting from bottom and wrapping your way around the right-hand side and moving upwards. Thus,

$(357)_{10}$ convert to $(101100101)_2$.

This method of conversion will work for converting to any non-decimal base. Just don't forget to include the first digit on the left corner, which is an indicator of the base. You can convert from base-ten (decimal) to any other base.

Conversion from other bases to Decimal

We write a number in decimal base as

$345 = 300 + 40 + 5 = 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$

Similarly, when a number is converted from any base to the decimal base then we write the number in that base in the expanded form and the result is the number in decimal form.

Ex.7 Convert $(1101)_2$ to decimal base

Sol. $(1101)_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 8 + 4 + 1 = 13$

So $(1101)_2 = (13)_{10}$

Ex.8 Convert the octal no 3456 in to decimal number.

Sol. $3456 = 6 + 5 \times 8 + 4 \times 8^2 + 3 \times 8^3$

$= 6 + 40 + 256 + 1536$

$= (1838)_{10}$

Ex.9 Convert $(1838)_{10}$ to octal.

Sol.

8	1838		
	229	-	6
	28	-	5
	3	-	4

$= (3456)_8$

Ex.10 What is the product of highest 3 digit number & highest 2 digit number of base 3 system?

(1) $(21000)_3$ **(2) $(22200)_3$** **(3) $(21222)_3$** **(4) $(21201)_3$** **(5) None**

Sol. The highest 3 digit & 2 digit numbers are 222 & 22

$222 = 2 + 2 \times 3 + 2 \times 3^2 = 26$

$22 = 2 + 2 \times 3 = 8$

$\therefore$ Product $= 26 \times 8 = 208$

Convert back to base

$(21201)_3$

3	208		
	69	-	1
	23	-	0
	7	-	2
	2	-	1

Answer: (4)

Ex.11 What is the remainder, if $24^{29} + 34^{29}$ is divided by 29?

Sol. $a^n + b^n$ is always divisible by a + b, if is odd.

$\therefore$ $24^{29} + 34^{29}$ is always divisible by 24 + 34 = 58.

So, it is always divisible by 29. So, the remainder is **0.**

Ex.12 What is the remainder, if 12^{243} is divided by 10?

Sol. 12^{243}

12	$\rightarrow$	2
12^2	$\rightarrow$	4
12^3	$\rightarrow$	8
12^4	$\rightarrow$	6
12^5	$\rightarrow$	2

The remainder repeats after every 4^{th} power.

So, the required answer is the remainder of 12^3 is divided by 10. i.e. **8**

Ex.13 What is the value of $(FBA)_{16}$ in binary system?

Sol. A = 10, B = 11, F = 15

Since $2^4 = 16$,

While converting each digit of the decimal, can be written as 4 digit binary no:

A = 1010, B = 1011, F = 1111

$(FBA)_{10} = (111110111010)_2$

Ex.14 Convert $(721)_8$ to binary.

Sol. Since $2^3 = 8$, write each digit of octal no. as 3 digits of binary which gives equivalent value.

7 = 111, 2 = 010, 1 = 001

$\therefore$ $(721)_8 = (111\ 010\ 001)_2$