Sequence & Series

A set of numbers whose domain is a real number is called a ***SEQUENCE*** and sum of the sequence is called a ***SERIES***. If $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_n + \ldots$ is a series.

Those sequences whose terms follow certain patterns are called ***progressions***.

For example 1, 4, 7, 10, 13 …….
7, 4, 1, – 2, – 5………
1, 2, 4, 8, 16………
8, 4, 2, 1, ½……….

Also if $f(n) = n^2$ is a sequence, then $f(1) = (1)^2 = 1$, $f(2) = 2^2 = 4$, $f(3) = (3)^2 = 9$
$f(10) = 10^2 = 100$ and so on.

The n^{th} term of a sequence is usually denoted by T_n

Thus T_1 = first term, T_2 = second term, T_{10} = tenth term and so on.

There are three different progressions

- **Arithmetic Progression (A.P)**
- **Geometric Progression (G.P)**
- **Harmonic Progression (H.P)**

Arithmetic Progression (A.P.)

It is a series in which any two consecutive terms have common difference and next term can be derived by adding that common difference in the previous term.

Therefore $T_{n+1} - T_n$ = constant and called common difference (d) for all $n \in N$.

Examples:

1. 1, 4, 7, 10, ……. is an A. P. whose first term is 1 and the common difference is
$d = (4 - 1) = (7 - 4) = (10 - 7) = 3$.

2. 11, 7, 3, – 1 …… is an A. P. whose first term is 11 and the common difference
$d = 7 - 11 = 3 - 7, = -1 - 3 = -4$.

If in an A. P. **a** = first term,
d = common difference = $T_n - T_{n-1}$
T_n = nth term

(Thus T_1 = first term, T_2 = second term, T_{10} tenth term and so on.)

l = last term,
S_n = Sum of the n terms.

Then a, a + d, a + 2d, a + 3d,... are in A.P.

n^{th} term of an A.P.

The n^{th} term of an A.P is given by the formula

$$T_n = a + (n - 1) d$$

Note: If the last term of the A.P. consisting of n terms be ℓ , then

$$\ell = a + (n - 1) d$$

Sum of n terms of an A.P

The sum of first n terms of an AP is usually denoted by S_n and is given by the following formula:

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$$

Where '*l*' is the last term of the series.

TIP

In an A.P of n terms, the sum of $T_r + T_{n-r+1}$ is always same for that A.P.

Ex.1 ***Find the series whose n^{th} term is $\frac{3n-1}{2}$. Is it an A. P. series? If yes, find 101^{st} term.***

Sol. Putting 1, 2, 3, 4…. We get T_1, T_2, T_3, T_4…………..

$= 1, \frac{5}{2}, 4, \frac{11}{2} \ldots$

$d_1 = \frac{3}{2}, d_2 = \frac{3}{2}, d_3 = \frac{3}{2}$

As the common differences are equal

$\therefore$ The series is an A.P.

$T_{101} = a + 100d = 1 + 100 \times \frac{3}{2} = 1 + 150 = 151.$

Ex.2 ***Find 8^{th}, 12^{th} and 16^{th} terms of the series; – 6, – 2, 2, 6, 10, 14, 18…***

Sol. Here a = – 6 and d = – 2 – (– 6) = 4.

$\therefore T_8 = -6 + 7 \times 4 = 22$ $[T_8 = a + 7d]$

$T_{12} = a + 11d = -6 + 11 \times 4 = 38$ $[T_{12} = a + 11d]$

$T_{16} = a + 15d = -6 + 15 \times 4 = 54$ $[T_{16} = a + 15d]$

FACT

if m times m^{th} term of an A.P. is equal to n times n^{th} term of same A.P. then $(m + n)^{th}$ term will be zero.

i .e $mT_m = nT_n$

$\Rightarrow T_{m+n} = 0$

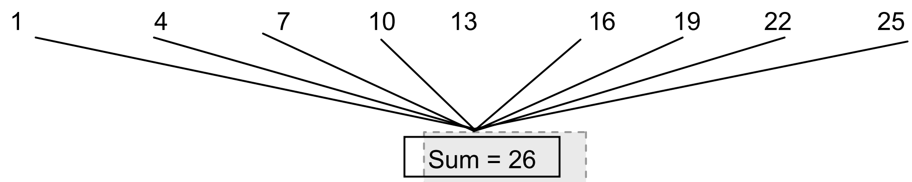
Properties of an AP

I. If each term of an AP is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

For example: For A.P. 3, 5, 7, 9, 11…

If you add constant let us say 1 in each term, you get	4, 6, 8, 10, 12......	This is an A.P. with common difference 2
If you multiply by a constant let us say 2 each term, you get	6, 10, 14, 18, 22…..	Again this is an A.P. of common difference 4

II. In an AP, the sum of terms equidistant from the beginning and end is always same and equal to the sum of first and last terms as shown in example below.



III. Three numbers in AP are taken as a – d, a, a + d.
For 4 numbers in AP are taken as a – 3d, a – d, a + d, a + 3d.
For 5 numbers in AP are taken as a – 2d, a – d, a, a + d, a + 2d.

IV. Three numbers a, b, c are in A.P. if and only if
2b = a + c.
or $b = \frac{a+c}{2}$ and b is called Arithmetic mean of a & c

Ex.3 ***The sum of three numbers in A.P. is – 3, and their product is 8. Find the numbers.***

Sol. Let the numbers be (a – d), a, (a + d). Then,
Sum = – 3 $\Rightarrow$ (a – d) + a + (a + d) = – 3 $\Rightarrow$ 3a = – 3 $\Rightarrow$ a = – 1
Product = 8
$\Rightarrow (a - d)(a)(a + d) = 8$
$\Rightarrow a(a^2 - d^2) = 8$
$\Rightarrow (-1)(1 - d^2) = 8$
$\Rightarrow d^2 = 9$
$\Rightarrow d = \pm 3$
If d = 3, the numbers are – 4, – 1, 2. If d = – 3, the numbers are 2, – 1, – 4.
Thus, the numbers are – 4, – 1, 2 or 2, – 1, – 4.

Ex.4 ***A student purchases a pen for Rs. 100. At the end of 8 years, it is valued at Rs. 20. Assuming that the yearly depreciation is constant. Find the annual depreciation.***

Sol. Original cost of pen = Rs. 100
Let D be the annual depreciation.
$\therefore$ Price after one year = 100 – D = T_1 = a (say)
$\therefore$ Price after eight years = T_8 = a + 7 (– D) = a – 7D
= 100 – D – 7D = 100 – 8D
By the given condition 100 – 8D = 20
8D = 80 $\therefore$ D = 10.
Hence annual depreciation = Rs. 10.

Geometric Progression

A series in which each preceding term is formed by multiplying it by a constant factor is called a Geometric Progression G. P. The constant factor is called the common ratio and is formed by dividing any term by the term which precedes it.
In other words, a sequence, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression

If $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$.

The General form of a G. P. with n terms is a, ar, ar^2,…ar^{n-1}

Thus if **a** = the first term

r = the common ratio,

$\mathbf{T_n}$ = nth term and

$\mathbf{S_n}$ = sum of n terms

General term of GP = $\mathbf{T_n = ar^{n-1}}$

Ex.5 ***Find the 9th term and the general term of the progression.***

$$\mathbf{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\ldots\ldots\ldots}$$

Sol. The given sequence is clearly a G. P. with first term a = 1 and common ratio = r = $\left(-\frac{1}{2}\right)$.

Now $T_9 = ar^8 = 1\left(-\frac{1}{2}\right)^8 = \frac{1}{2^8} = \frac{1}{256}$ and $T_n = ar^{n-1} = 1.\left(-\frac{1}{2}\right)^{n-1}$

$= (-1)^{n-1} \cdot \frac{1}{2^{n-1}}$

<u>**Sum of n terms of a G.P:**</u>

$$\mathbf{S_n} = \frac{a(r^n - 1)}{r - 1} \quad \textbf{where r > 1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{where r < 1}$$

$$\mathbf{S_n = an} \quad \textbf{where r = 1}$$

Sum of infinite G.P:

If a G.P. has **infinite terms** and **–1 < r < 1 or $|x|$ < 1**, then sum of infinite G.P is $\mathbf{S_\infty = \frac{a}{1-r}}$.

Ex.6 ***The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the number of the grains for subsequent squares. How many grains would have to be given to inventor? (There are 64 squares in the chess board).***

Sol. Required number of grains

$= 1 + 2 + 2^2 + 2^3 + \ldots\ldots$ To 64 terms $= 1\left(\frac{2^{64} - 1}{2 - 1}\right) = 2^{64} - 1$.

Recurring Decimals as Fractions.

If in the decimal representation a number occurs again and again, then we place a dot (.) on the number and read it as that the number is recurring.

e.g., 0.5 (read as decimal 5 recurring).

This mean $0.\overline{5}$ = 0.55555…….∞

$0.4\overline{7} = 0.477777……\infty$

These can be converted into fractions as shown in the example given below

Ex.7 ***Find the value in fractions which is same as of $0.4\overline{37}$***

Sol. We have $0.4\overline{37}$ = .4373737……..∞

= 0.4 + 0.037 + 0.00037 + 0.0000037 + …..∞

$= \frac{4}{10} + \frac{37}{10^3} + \frac{37}{10^5} + \frac{37}{10^7}$ ………∞

$= \frac{4}{10} + \frac{37/10^3}{1 - \frac{1}{10^2}}$ $\left[\text{Here } a = \frac{37}{10^3}; r = \frac{1}{10^2}\right]$

$= \frac{4}{10} + \frac{37}{1000} \times \frac{100}{99} = \frac{4}{10} + \frac{37}{990} = \frac{396+37}{990} = \frac{433}{990}$

Properties of G.P.

I. If each term of a GP is multiplied or divided by the same non-zero quantity, then the resulting sequence is also a GP.

For example: For G.P. is 2, 4, 8, 16, 32…

If you multiply each term by constant let say 2,you get	4, 8, 16, 32, 64..	This is a G.P.
If you divide each term by constant let say 2,you get	1, 2, 4, 8, 16 ..	This is a G.P.

II. **SELECTION OF TERMS IN G.P.**

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner :

No. of terms	***Terms***	***Common ratio***
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

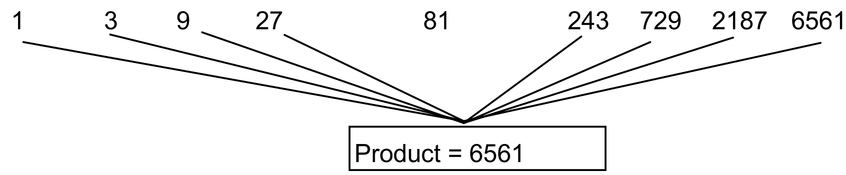
If the product of the numbers is not given, then the numbers are taken as a, ar, ar^2, ar^3, ….

III. Three non-zero numbers a, b, c are in G.P. if and only if

$b^2 = ac$ or $b = \sqrt{ac}$

b is called the geometric mean of a & c

IV. In a GP, the product of terms equidistant from the beginning and end is always same and equal to the product of first and last terms as shown in the next example.



Harmonic Progression (H.P.)

- A series of quantities is said to be in a harmonic progression when their reciprocals are in arithmetic progression.
- **e.g.** $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, …. and $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$, ….. are in HP as their reciprocals
- 3, 5, 7, …, and a, a + d, a + 2d…….. are in AP.

n^{th} term of HP

- Find the nth term of the corresponding AP and then take its reciprocal.
- If the HP be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$ …
- Then the corresponding AP is a, a + d, a + 2d, ……
- T_n of the AP is a + (n – 1) d
- T_{nth} of the HP is …… $\frac{1}{a+(n-1)d}$
- In order to solve a question on HP, one should form the corresponding AP.

A comparison between AP and GP

Description	AP	GP
Principal Characteristic	Common Difference (d)	Common Ratio (r)
n^{th} Term	$T_n = a + (n - 1) d$	$T_n = ar^{(n-1)}$
Mean	A = (a + b) / 2	$G = (ab)^{1/2}$
Sum of First n Terms	$S_n = n/2 [2a + (n - 1) d] = n/2 [a + \ell]$	$S_n = a (1 - r^n) / (1 - r)$
'm'th mean	a+ [m (b – a) / (n + 1)]	$a (b / a)^{m/ (n+1)}$

Arithmetic – Geometric progression

$a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 +$ ……………. Is the form of Arithmetic geometric progression (A.G.P). One part of the series is in Arithmetic progression and other part is a Geometric progression.

The sum of n terms series is $S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)}$.

The infinite term series sum is $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Arithmetic geometric series can be solved as explained in the example below:

Relation between AM, GM and HM:

For two positive numbers a and b

A = Arithmetic mean = $\frac{a+b}{2}$

G = Geometric Mean = $\sqrt{ab}$

H = Harmonic Mean = $\frac{2ab}{a+b}$

Multiplying A and H, we get AH = $\frac{a+b}{2} \times \frac{2ab}{a+b}$ = ab = G^2

This mean A, G, H are in G.P.

Verifying for numbers 1, 2

AM = $\frac{1+2}{2}$ = 1.5 = $\frac{3}{2}$, GM = $\sqrt{2}$ and HM = $\frac{2 \times 1 \times 2}{3} = \frac{4}{3}$

Hence **AM $\geq$ GM $\geq$ HM**

G^2 = 2, and **AH** = $\frac{3}{2} \times \frac{4}{3}$ = 2

Hence **G^2 = AH**

> ***Do you know?***
> AM $\geq$ GM $\geq$ HM (always for positive numbers) and G^2 = AH

Toolkit

$$\sum n = 1 + 2 + 3 + \ldots\ldots. n = \frac{n(n+1)}{2}$$

$$\sum n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots. n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots. n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Ex.8 ***Find the sum of 1 + 2x + 3x² + 4x³ + … ∞***

Sol. The given series in an arithmetic-geometric series whose corresponding A.P. and G.P. are 1, 2, 3, 4,… and 1, x, x^2, x^3, … respectively. The common ratio of the G.P. is x. Let S_∞ denote the sum of the given series.

Then, $S_\infty = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$ ………(i)

$\Rightarrow x S_\infty = x + 2x^2 + 3x^3 + \ldots \infty$ ………(ii)

Subtracting (ii) from (i), we get

$S_\infty - x S_\infty = 1 + [x + x^2 + x^3 + \ldots \infty]$

$\Rightarrow \quad S_\infty (1 - x) = 1 + \frac{x}{1-x}$

$\Rightarrow \quad S_\infty = \frac{1}{1-x} + \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2}$

Ex.9 If the first item of an A.P is 12, and 6^{th} term is 27. What is the sum of first 10 terms?

Sol. $a = 12, t_6 = a + 5d = 27 \Rightarrow d = 3$

$\therefore S_{10} = \frac{10}{2}[2 \times 12 + (10 - 1)3] = 255.$

Ex.10 If the fourth & sixth terms of an A.P are 6.5 and 9.5. What is the 9^{th} term of that A.P?

Sol. $a + 3d = 6.5$ & $a + 5d = 9.5$

$\Rightarrow a = 2,$ & $d = 1.5$

$\therefore \mathbf{t_9} = a + 8d = \mathbf{14}$

Ex.11 What is the arithmetic mean of first 20 terms of an A.P. whose first term is 5 and 4^{th} term is 20?

Sol. $a = 5, \mathbf{t_4} = a + 3d = 20 \Rightarrow d = 5$

A.M is the middle number = average of 10^{th} & 11^{th} number = $\frac{50+55}{2} = 52.5$

(or) A.M = $\frac{Sn}{n} = \frac{1}{2}[2a + (n-1)d]$, where a = 5, n = 20, d = 5 $\Rightarrow$ A.M = **52.5**

Ex.12 The first term of a G.P is half of its fourth term. What is the 12^{th} term of that G.P, if its sixth term is 6

Sol. $t_1 = \frac{1}{2}t_4$

$\Rightarrow a = \frac{1}{2}ar^3 \Rightarrow r^3 = 2$

$t_6 = ar^5 = 6$

$t_{12} = ar^{11} = ar^5 \times r^6 = 6(2)^2 = \mathbf{24}$

Ex.13 If the first and fifth terms of a G.P are 2 and 162. What is the sum of these five terms?

Sol. $a = 2$

$ar^4 = 162 \Rightarrow r = 3$

$S_5 = \frac{2(3^5 - 1)}{3 - 1} = 242$

Ex.14 What is the value of $r + 3r^2 + 5r^3 + - - - - -$

Sol. Assume $S = r + 3r^2 + 5r^3 + - - - - -$ ….(1)

$r \times S = r^2 + 3r^3 + 5r^4 + ...$ ….(2)

(1) – (2)

$s(1-r) = r + 2r^2 + 2r^3 + ...$

$= r + \frac{2r^2}{1-r} = \frac{1+r^2}{1-r} \Rightarrow s = \frac{1+r^2}{(1-r)^2}$

Ex.15 The first term of a G.P. 2 and common ratio is 3. If the sum of first n terms of this G.P is greater than 243 then the minimum value of 'n' is

Sol. $\frac{a(r^n - 1)}{r - 1} > 243$

$\Rightarrow \frac{2(3^n - 1)}{3 - 1} > 243$

$\Rightarrow 3^n > 244$

$\Rightarrow n > 5$ So, min possible value of n is **6.**

Ex.16 $\frac{1}{2\times 5} + \frac{1}{3\times 6} + \frac{1}{4\times 7} + ------- + \frac{1}{11\times 14}$ is

Sol. $= \frac{1}{3}\left[\frac{5-2}{2\times 5} + \frac{6-3}{3\times 6} + \frac{7-4}{4\times 7} + ----- + \frac{14-11}{11\times 14}\right]$

$= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + ----- + \frac{1}{11} - \frac{1}{14}\right]$

$= \frac{1}{3}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} - \frac{1}{13} - \frac{1}{14}\right]$

$= \frac{1}{3}\left[\frac{546 + 364 + 273 - 91 - 84 - 78}{12\times 13\times 7}\right] = \frac{155}{546}$

Ex.17 $a_n = 2^n + 1$ **then** $(a_1 + a_2 + - - - - + a_{20}) - a_{21}$ **is:**

Sol. $a_1 + a_2 + - - - - + a_{20} = 2^1 + 1 + 2^2 + 1 + - - - - - + 2^{20} + 1$

$= 2(2^{20} - 1) + 20$

$= 2^{21} + 18$

$\therefore$ Ans = **18**