Quadratic Equations

(Polynomials, Equations, Remainder theorem)

Polynomials

Definition: *Let $a_0, a_1, a_2, \ldots.. a_n$ be real numbers and x is a real variable. Then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.. + a_n x^n$ is called a real polynomial of real variable x with coefficients $a_0, a_1, a_2, \ldots.. a_n$.*

Examples: $x^3 + 4x^2 - 3$ is a polynomial

Degree of a Polynomial:

Degree of a polynomial is the highest power of the variable in the polynomial.

Example:

Degree of $3x^3 + 4x^2 - 3$ is 3 as the maximum power of the variable x is 3.

On the basis of degree, the polynomials are classified as Linear (Degree 1), Quadratic (degree 2), Cubical (Degree 3), bi-Quadratic (degree 4), and so on.

Remainder Theorem

If any polynomial f(x) is divided by (x – a) then f(a) is the remainder.

For example,

$f(x) = x^2 - 5x + 7 = 0$ is divided by x – 2. What is the remainder?

$R = f(2) = 2^2 - 5 \times 2 + 7 = 1$.

Factor Theorem

If (x – a) is a factor of f(x), then remainder f(a) = 0. (Or) if f(a) = 0, then (x – a) is a factor of f(x),

For example,

When $f(x) = x^2 - 5x + 6 = 0$ is divided by x – 2, the remainder f(2) is zero which shows that x – 2 is the factor of f(x)

General Theory of Equations

An equation is the form of a polynomial which has been equated to some real value.

For example:

$2x + 5 = 0$, $x^2 - 2x + 5 = 7$, $2x^2 - 5x^2 + 1 = 2x + 5$ etc. are polynomial equations.

Root or Zero of a polynomial equation:

If f (x) = 0 is a polynomial equation and $f(\alpha) = 0$, then α is called a root or zero of the polynomial equation f(x) = 0.

Linear Equation

Linear Equation with one variable:

A linear equation is 1^{st} degree equation. It has only one root. Its general form is a x + b = 0 and root is $-\frac{b}{a}$.

> **TIP**
> If we plot the graph of y = f(x) A linear equation is 1^{st} degree equation. It has only one root. Its general form is a x + b = 0 and root is $-\frac{b}{a}$.

Linear equation with two variables:

It is a first degree equation with two variables.

Ex: $2x + 3y = 0$.

We need two equations to find the values of x and y.

If there are n variables in an equation, we need n equations to find the values of the variables uniquely.

Some times, even the number of equations are equal to the number of variables, we cannot find the values of x and y uniquely.

For example, $3x + 5y = 6$

$6x + 10y = 12$

These are two equations, but both are one and the same. So different values of x and y satisfy the equation and there is no unique solution. It will has infinite number of solutions.

The number of solutions is clearly described below for the set of equations with 2 variables.

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

These equations can be

1. Inconsistent means have no solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

2. Consistent and has infinitely many solutions if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

3. Consistent and have unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Quadratic Equation

Quadratic Equation in "x" is one in which the highest power of "x" is 2. The equation is generally satisfied by two values of "x".

The quadratic form is generally represented by $ax^2 + bx + c = 0$ where $a \neq 0$, and a, b, c are constants.

For Example:

$$x^2 - 6x + 4 = 0$$
$$3x^2 + 7x - 2 = 0$$

A quadratic equation in one variable has two and only two roots, which are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

FUNDA

Imaginary roots are always unequal and conjugate of each other. i.e. If one root is 3 + 5i the other will be 3 – 5i.

Nature of Roots

The two roots of any quadratic equation always depend on the value of $b^2 - 4ac$ called discriminant (D).

$D > 0$	Real and unequal roots
$D = 0$	Real and equal
$D < 0$	Imaginary and unequal

TIPS

1. If roots of given equation are equal in magnitude but opposite in sign, then b = 0 & vice versa.

2. If roots of given equation are reciprocal of each other, then c = a.

Sum and Product of roots:

If α and β are the two roots of $ax^2 + bx + c = 0$,

Then **sum of roots** = $\alpha + \beta = -\frac{b}{a} = -\frac{(x \text{ coefficient})}{(x^2 \text{ coefficient})}$

And **product of roots** = $\alpha\beta = \frac{c}{a} = \frac{\text{constant}}{x^2 \text{ coefficient}}$

Formation of equation from roots:

If α and β are the roots of any quadratic equation then that equation can be written in the form

$$X^2 - (\alpha + \beta)X + \alpha\beta = 0$$

i.e. X^2 – (sum of the roots) X + Product of the roots = 0

Some Important results

If x_1, x_2 *are the roots of the equation* $f(x) = ax^2 + bx + c = 0$, *then,*

1. The *equation whose roots are* equal in magnitude and opposite in sign to that of f(x) i.e., $-x_1, -x_2$ *is* $f(-x) = 0$ *i.e.,* $ax^2 - bx + c = 0$.
2. The equation whose roots are reciprocals to that of f(x) i.e., $\frac{1}{x_1}$ and $\frac{1}{x_2}$ is $f(1/x) = 0$. i.e. $cx^2 + bx + a = 0$.
3. The equation whose roots are k more/less than that of f(x) i.e., $x_1 \pm k$, $x_2 \pm k$ *is* $f(x \mp k) = 0$. i.e. $a(x \mp k)^2 + b(x \mp k) + c = 0$.
4. The equation whose roots are k times to that of f(x) i.e., kx_1, kx_2, *is* $f(x/k) = 0$. i.e. $ax^2 + kbx + k^2c = 0$
5. The equation whose roots are x_1^k, x_2^k *is* $f(\sqrt[k]{x}) = 0$. i.e. $a(\sqrt[k]{x})^2 + b(\sqrt[k]{x}) + c = 0$.

Do you know?

1. If one of the roots of a quadratic equation is $k_1 + \sqrt{k_2}$, then the other root will be $k_1 - \sqrt{k_2}$ & vice versa.
2. If one of the roots be $(m + in)$, then the other root will be $(m - in)$ & vice versa.

Relation between roots and coefficient of an equation

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the n roots of the equation:

$$\mathbf{a_0x^n + a_1x^{n-1} + a_2x^{n-1} \ldots + a_{n-1}x + a_n = 0}$$

Then we have the following relations:

Sum of the roots taken one at a time $(\alpha_1 + \alpha_2 + \ldots + \alpha_n) = -(a_1/a_0)$.

Sum of the roots taken two at a time $(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \ldots \alpha_n\alpha_1) = (a_2/a_0)$

Sum of the roots taken three at a time $(\alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots + \alpha_n\alpha_1\alpha_2) = -(a_3/a_0)$

Product of the roots = $(\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n) = \{(-1)^n a_n / a_0\}$.

E.g. **Polynomial equation** $\mathbf{ax^4 + bx^3 + cx^2 + dx + e} = 0$

Sum of the roots = $-\frac{b}{a}$.

Sum of the roots taking two at a time = $\frac{c}{a}$

Sum of the roots taking three at a time = $-\frac{d}{a}$

and Product of all the roots = $\frac{e}{a}$.

Maximum and Minimum value of a Quadratic equation

The quadratic equation $ax^2 + bx + c = 0$ will have maximum or minimum value at $x = -b/2a$. If $a < 0$, it has maximum value and if $a > 0$, it has minimum value.

The maximum or minimum value is given by $\frac{4ac - b^2}{4a}$.

Ex.1 Solve for x: $\frac{3x+4}{x-3} = \frac{2}{3}$

Sol. $3(3x + 4) = 2(x - 3)$

$\Rightarrow 7x = -18 \quad \Rightarrow x = -\frac{18}{7}$

Ex.2 A and B went to a hotel paid Rs. 84 for 3 plates of Idli and 5 plates of Dosa. Where as B took 5 plates of Idli and 3 plates of Dosa and paid Rs. 76. What is the cost of one plate of Idli.

Sol. $3I + 5D = 84$ ……….(1)

$5I + 3D = 76$ ………(2)

Equation (1) × 3 - equation (2) × 5, we get

$16I = 128$

$\Rightarrow I = 8$

Each plate of Idli cost Rs. 8.

Ex.3 Find the values of x and y from the equations $\frac{4}{\sqrt{x}} - \frac{3}{\sqrt{y}} = 1$ **and** $\frac{1}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 3.5$.

Sol. Take $\frac{1}{\sqrt{x}} = a$, $\frac{1}{\sqrt{y}} = b$

Then, the equations will become

$4a - 3b = 1$ ……..(1)

And $a + 9b = 3.5$ ………(2)

(1) × 3 + (2)

$\Rightarrow a = \frac{1}{2}$ and by substituting a in either (1) or (2), we can get $b = \frac{1}{3}$.

$\therefore a = \frac{1}{\sqrt{x}} = \frac{1}{2} \Rightarrow x = 4$ and $b = \frac{1}{\sqrt{y}} = \frac{1}{3} \Rightarrow y = 9$

Ex.4 Find x and y from $\frac{1}{x-y} + \frac{5}{3x-2y} = 2$, **and** $\frac{15}{3x-2y} - \frac{2}{x-y} = 1$.

Sol. Take $\frac{1}{x-y} = a$ and $\frac{1}{3x-2y} = b$

$\therefore a + 5b = 2$ ……(1)

and $15b - 2a = 1$ ……(2)

(1) × 2 + (2) $\Rightarrow b = \frac{1}{5}$

So, a = 1.

$\therefore$ x – y = $\frac{1}{a}$ = 1 ……..(3)

and 3x – 2y = $\frac{1}{b}$ = 5 ……(4)

(3) × (3) – (4) gives Y = 2 and X = 3.

Ex.5 Aman won a competition and so he got some prize money. He gave Rs. 2000 less than the half of prize money to his son and Rs. 1000 more than the two third of the remaining to his daughter. If both they got same amount, what is the prize money Aman got?

Sol. Assume Aman got x rupees.

He gave $\frac{x}{2}$ – 2000 to his son.

And $\frac{2}{3}\left(\frac{x}{2}+2000\right)$ + 1000 to his daughter.

$\therefore$ $\frac{x}{2}$ – 2000 = $\frac{x}{3}$ + $\frac{7000}{3}$

$\Rightarrow$ x = Rs. 26000

Ex.6 How many non negative integer pairs (x, y) satisfy the equation, 3x + 4y = 21?

Sol. Since x and y are non negative integers.

Start from x = 0.

If x = 0 or 2, y cannot be integer.

For x = 3, y = 3.

And for x = 7, y = 0.

These two pairs only satisfy the given equation.

Ex.7 If (x – 2) is a factor of $x^3 - 3x^2 + px + 4$. Find the value of p.

Sol. Since (x – 2) is a factor, f(2) = 0.

$\therefore 2^3 - 3(2^2) + (2)p + 4 = 0$

$\Rightarrow$ p = 0

Ex.8 When $x^3 - 7x^2 + 3x - P$ is divided by x + 3, the remainder is 4, then what is the value of P ?

Sol. f(– 3) = 4

$\therefore (-3)^3 - 7(-3)^2 + 3(-3) - P = 4$

P = – 103.

Ex.9 If (x – 1) is the HCF of $(x^3 - px^2 + qx - 3)$ and $(x^3 - 2x^2 + px + 2)$. What is the value of ‘q’?

Sol. Since (x – 1) is HCF, it is a factor for both the polynomials.

$\therefore 1^3 - p(1)^2 + q(1) - 3 = 0$ $\Rightarrow$ – p + q = 2

And $1^3 - 2(1^2) + p(1) + 2 = 0$

p = – 1

$\therefore$ q = 1

Ex.10 Find the roots of the quadratic equation $x^2 - x - 12 = 0$.

Sol. $x^2 - x - 12 = 0$

$\Rightarrow x^2 - 4x + 3x - 12 = 0$

$x(x - 4) - 3(x - 4) = 0$

$(x - 4)(x - 3) = 0$

$\Rightarrow x = 4$ or 3. The roots are 4 and 3.

Ex.11 Find the roots of the quadratic equation $x^2 - 8x + 5 = 0$.

Sol. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = \frac{-(-8) \pm \sqrt{8^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{8 \pm \sqrt{44}}{2} = 4 \pm \sqrt{11}$ The roots are $4 + \sqrt{11}$ and $4 - \sqrt{11}$.

Ex.12 If 3 + 4i is a root of quadratic equation $x^2 - px + q = 0$. What is the value of pq? *(Given i is known as iota and $i^2 = -1$)*

Sol. If 3 + 4i is one root of a quadratic equation, 3 – 4i will be the other root.

(Imaginary roots exist in conjugate pairs)

Sum of roots = p = (3 + 4i) + (3 – 4i) $\Rightarrow$ p = 6

Product of the roots = q = (3 + 4i) (3 – 4i)

$\Rightarrow$ q = 25

$\therefore$ pq = 150.

Ex.13 If one root of a quadratic equation $x^2 - px + 8 = 0$ is square of the other, what is the value of p?

Sol. Let the roots of α, α^2.

Sum = $\alpha + \alpha^2 = p$

Product = $\alpha(\alpha^2) = 8$ $\Rightarrow \alpha = 2$

$\therefore p = 2 + 2^2 = 6$.

Ex.14 Describe the nature of the roots of the equation $x - \frac{1}{x} = 3$.

Sol. Given equation can be written as $x^2 - 3x - 1 = 0$.

Discriminent = $(-3)^2 - 4(1)(-1) = 13 > 0$.

So, roots are real and distinct.

Ex.15 If α, β are the roots of $x^2 - 7x + P = 0$, and $\alpha - \beta = 3$, then what is the value of P?

Sol. Sum = $\alpha + \beta = 7$ ……….(1)

Product = $\alpha\beta = P$ ……….(2)

Given, $\alpha - \beta = 3$ ………..(3)

(1) and (3) $\Rightarrow \alpha = 5, \beta = 2$

$\therefore$ P = 10 (from (2))

Ex.16 Form a quadratic equation, whose one of the roots is $2 + \sqrt{3}$.

Sol. If $2 + \sqrt{3}$ is one root, other root will be $2 - \sqrt{3}$.

$\therefore$ The equation is $x^2 - (2 + \sqrt{3} + 2 - \sqrt{3})x + (2 + \sqrt{3})(2 - \sqrt{3}) = 0$

$\Rightarrow x^2 - 4x + 1 = 0$

Ex.17 Find the values of x, which satisfy the equation $4^x - (10)\,2^x + 16 = 0$.

Sol. Assume $2^x = k$

$\therefore$ The given equation will become $k^2 - 10k + 16 = 0$.

$k^2 - 8k - 2k + 16 = 0$

$(k - 2)(k - 8) = 0$

$\Rightarrow k = 2$ or 8

$\therefore 2^x = 2$ or $2^x = 8$

$\therefore x = 1$ or 3.

Ex.18 What values of x satisfy the given equations $4k.x - x^2 = 4k^2$ and $(x - k)^2 + 3(x - k) - 7 = 0$?

Sol. The first equation can be written as $(x - 2k)^2 = 0$.

$\Rightarrow x = 2k \quad \Rightarrow k = \frac{x}{2}$

$\therefore$ The second equation will become

$$\left(x - \frac{x}{2}\right)^2 + 3\left(x - \frac{x}{2}\right) - 7 = 0$$

$x^2 + 6x + 28 = 0$

$$\therefore x = \frac{-6 \pm \sqrt{36 - 112}}{2}$$

$x = -3 \pm \sqrt{37}$.

Ex.19 If x_1 and x_2 are the roots of the equation $x^2 - 2x + 14 = 0$, what is the equation with roots $3x_1 - 2$ and $3x_2 - 2$?

Sol. The equation with roots $3x_1$ and $3x_2$ is $\left(\frac{x}{3}\right)^2 - 2\left(\frac{x}{3}\right) + 4 = 0$.

$\Rightarrow x^2 - 6x + 36 = 0$

The equation with roots $3x_1 - 2$ and $3x_2 - 2$ is

$(x + 2)^2 - 6(x + 2) + 36 = 0$

$\Rightarrow x^2 - 2x + 28 = 0$.

Ex.20 If α, β and γ are the roots of a cubic equation, $x^3 - 2x^2 + x - 5 = 0$, then what is the value of $\alpha^2 + \beta^2 + \gamma^2$?

Sol. Sum of roots $= \alpha + \beta + \gamma = 2$.

$\alpha\beta + \beta\gamma + \gamma\alpha = 1$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 2^2 - 2(1) = 2$.

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 2$.

Ex.21 Find the value of $\dfrac{1}{2-\dfrac{1}{3+\dfrac{1}{2-\dfrac{1}{3+\dfrac{1}{2-\dfrac{1}{3+....}}}}}}$.

Sol. Assume the given is x.

So it can be written as x = $\dfrac{1}{2-\dfrac{1}{3+x}}$

$\therefore$ x = $\dfrac{3+x}{5+2x}$

$\Rightarrow$ 5x + 2x^2 = 3 + x

2x^2 + 4x – 3 = 0

x = $\dfrac{-4 \pm \sqrt{16+24}}{4}$

x = – 1 ± $\sqrt{\dfrac{5}{2}}$.

But x cannot be negative.

So x = – 1 + $\sqrt{\dfrac{5}{2}}$.

Ex.22 What is the value of x^2 – 2x + 3, when the value of 2x^2 – 5x + 6 is minimum?

Sol. The value of 2x^2 – 5x + 6 will be minimum at x = $\dfrac{-(-5)}{2\times 2} = \dfrac{5}{4}$ $\left(x = \dfrac{-b}{2a}\right)$

$\therefore$ At x = $\dfrac{5}{4}$ x^2 – 2x + 3 = $\dfrac{33}{16}$.

Ex.23 What is the value of k, if the quadratic equation x^2 – 6x + k has only one real root?

Sol. If one root is imaginary, the other should be its conjugate, which is also imaginary.

So, the given equation has only one real root and there is no chance for the other root to be imaginary, so the roots should be equal.

$\therefore$ Discriminent = 0

(– 6)^2 – 4 × 1 × k = 0

$\Rightarrow$ k = 9.

Ex.24 What is the value of p – q, if the roots of the quadratic equation px^2 – (p + q)x + q = 0 are reciprocals?

Sol. Assume the roots be α, $\dfrac{1}{\alpha}$.

$\therefore$ Product = $\dfrac{q}{p}$ = 1 $\Rightarrow$ q = p $\therefore$ p – q = 0